

# Engineering Mechanics: Statics

**Engineering**  
**Mechanics:**  
**Statics**





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*LIBBY OSGOOD, GAYLA CAMERON, AND EMMA  
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# Contents







Welcome! This book is an introduction to engineering mechanics: statics, when acceleration is 0. Hopefully, this course will help you to see statics *everywhere* in the world - because it truly is *everywhere*! Concepts include:

- particles and rigid body equilibrium equations,
- free-body diagrams,
- distributed loads,
- shear and moment diagrams,
- trusses, method of joints and sections, &
- inertia.

This is the first of two courses to describe how objects move and the forces that cause motion. This combines math and physics fundamentals with real-world application. A structured problem-solving process is included, and by the end of the book, you should be able to recognize and describe motion all around you in your everyday life.

Chapter 1 contains the fundamental math and physics concepts including vectors, Pythagorean theorem, sine and cosine laws, dot product, Newton's laws, weight and mass, unit conversions, and the problem solving process.

Chapter 2 explains the difference between particles and rigid bodies and introduces free-body diagrams and equilibrium equations for particles.

Chapter 3 contains introductory rigid body concepts, including cross products, the right hand rule, torques/moments and couples, distributed loads and reaction/support forces.

Chapter 4 introduces free-body diagrams and equilibrium equations for rigid bodies, as well as external forces, frictional and impending motion.

Chapter 5 introduces trusses and two methods to solve truss systems: method of joints and method of sections.

Chapter 6 explains internal forces and breaks down shear/moment diagrams.

Chapter 7 introduces center of mass, mass moment of inertia, area moment of inertia, and the parallel axis theorem.

Appendix A has a reference list of open textbooks.

This book is a combination of many other open educational resources (OER) under similar creative commons licenses. This information is denoted inside a box like this. The links to the original source are very clearly included

so you can go to those books to see what they say on other topics and try out their sample problems.

#### Key Takeaways

Most sections have a 'key takeaway' that includes text from me (Libby), containing the most important part of the section (*Basically*), where it occurs in the real world (*Application*), and what part of this course it connects to - the why are we learning this (*Looking ahead*).

The last section of each chapter includes examples that were submitted by former ENGN 1230 statics students to help you learn. Statics and most engineering courses are 'team sports'. I recommend finding a few study partners to struggle through the homework and study for the tests together.

This book is a good start to helping you learn, but ultimately it's up to you. Complete **every** homework and go to **every** class. But that's not enough. Look at the solutions that are posted and practice. You'll get out of it what you put in, and statics can be fun. It's how engineers apply physics concepts to the real world. I hope you learn to love learning engineering as much as I enjoy teaching it. Hopefully, you'll get a sense of the wonder of engineering through this book.

Book cover image by [malinoh](#) from [Pixabay](#)



# CHAPTER 1: FUNDAMENTAL CONCEPTS

## Static vs Dynamic Motion

Before we start, what is the difference between **static** and **dynamic**? Static problems are all problems where there is no acceleration. As you're driving down the road and you're cruising along at a constant velocity, that is a static problem. As soon as you start to slow down for a stop light or speed up, you are in dynamic motion, and that's much more complicated. For this course, we will only consider problems where there is **no motion** (such as the 'static' is used in the English language), or **constant velocity**. Be prepared to give many examples of static versus dynamic, because before you can solve a problem, you have to know what type of problem it is!

## Introduction to Chapter 1: Fundamental Concepts

This chapter has a lot of concepts from math and physics that are necessary for you to understand before we can apply them in engineering contexts. It's kind of like: before you can write an essay to express your opinion, you need to know how to write the a, b, c's and what each word means. Here, you need to know how to compute a cross product before you can calculate how much Torque is created from a force.

Some of this might be new. Some of this might be familiar, but we might be applying it in a different way. (Such as calculating torque - this is not what you learned in high school physics!) Some of it might feel new, so practice practice practice!

Here are the sections in this Chapter:

- 1.1 Math & Physics Important Concepts (Mass, Weight, Slugs, Trig, Units, Conversions, Scalar, Vector, Newton's Laws)
- 1.2 XYZ Coordinate Frame
- 1.3 Vectors
- 1.4 Dot Product
- 1.5 Cross Products
- 1.6 Torque/Moment
- 1.7 Problem Solving Process
- 1.8 Student examples

Here are the key equations and concepts you will learn in this chapter

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# Ch1 Equations

## Vector Notation

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\vec{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad \underline{A} = |A| \underline{\hat{a}}$$

## Magnitude (3d)

$$|\vec{A}| = \sqrt{(\bar{A}_x)^2 + (\bar{A}_y)^2 + (\bar{A}_z)^2}$$

## Unit Vector (3d)

$$\underline{\hat{a}} = \frac{\underline{A}}{|A|} = \frac{\bar{A}_x \hat{i} + \bar{A}_y \hat{j} + \bar{A}_z \hat{k}}{\sqrt{(\bar{A}_x)^2 + (\bar{A}_y)^2 + (\bar{A}_z)^2}}$$

Direction (2d)  $\theta = \tan^{-1} \frac{\bar{A}_y}{\bar{A}_x}$

Static ( $\underline{a} = 0$ ) & Dynamic ( $\underline{a} \neq 0$ )

$$F = mg$$

$$g = 32.2 \text{ ft/s}^2$$

$$g = 9.81 \text{ m/s}^2$$

Mass [=] kg or slug

$$\text{Slug} = \left( \frac{\text{lb} \cdot \text{s}^2}{\text{ft}} \right)$$



$$\text{SOH: } \sin \theta = \text{Opp} / \text{Hyp}$$

$$\text{CAH: } \cos \theta = \text{Adj} / \text{Hyp}$$

$$\text{TOA: } \tan \theta = \text{Opp} / \text{Adj}$$

## Moment / Torque

Scalar:  $M = |r| |F| \sin \Theta$

Vector:  $\underline{M} = \underline{r} \times \underline{F}$

	English	SI
Force	Lb	4.448 N
Mass	slug	14.59 kg
Length	Ft	0.3048 m

## Dot Product

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$$= |\vec{A}| |\vec{B}| \cos \theta$$

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|}$$

## Cross Product

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (a_2 b_3 - b_2 a_3) \hat{i} - (b_3 a_1 - a_3 b_1) \hat{j} + (a_1 b_2 - b_1 a_2) \hat{k}$$

$$|\vec{A} \times \vec{B}| = AB \sin \theta,$$

# 1.1 Preparatory Concepts

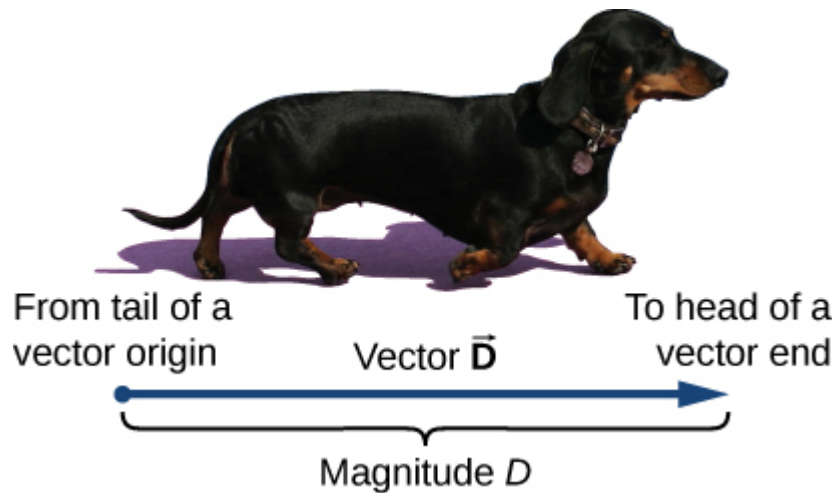
## 1.1.1 Scalar vs. Vector

Many familiar physical quantities can be specified completely by giving a single number and the appropriate unit. For example, “a class period lasts 50 min” or “the gas tank in my car holds 65 L” or “the distance between two posts is 100 m.” A physical quantity that can be specified completely in this manner is called a **scalar quantity**. Scalar is a synonym of “number.” Time, mass, distance, length, volume, temperature, and energy are examples of **scalar** quantities.

Scalar quantities that have the same physical units can be added or subtracted according to the usual rules of algebra for numbers. For example, a class ending 10 min earlier than 50 min lasts  $(50 \text{ min} - 10 \text{ min}) = 40 \text{ min}$ . Similarly, a 60-cal serving of corn followed by a 200-cal serving of donuts gives  $(60 \text{ cal} + 200 \text{ cal}) = 260 \text{ cal}$  of energy. When we multiply a scalar quantity by a number, we obtain the same scalar quantity but with a larger (or smaller) value. For example, if yesterday’s breakfast had 200 cal of energy and today’s breakfast has four times as much energy as it had yesterday, then today’s breakfast has  $4(200 \text{ cal}) = 800 \text{ cal}$  of energy. Two scalar quantities can also be multiplied or divided by each other to form a derived scalar quantity. For example, if a train covers a distance of 100 km in 1.0 h, its speed is  $100.0 \text{ km}/1.0 \text{ h} = 27.8 \text{ m/s}$ , where the speed is a derived scalar quantity obtained by dividing distance by time.

Many physical quantities, however, cannot be described completely by just a single number of physical units. For example, when the U.S. Coast Guard dispatches a ship or a helicopter for a rescue mission, the rescue team must know not only the distance to the distress signal, but also the direction from which the signal is coming so they can get to its origin as quickly as possible. Physical quantities specified completely by giving a number of units (magnitude) and a direction are called **vector quantities**. Examples of vector quantities include displacement, velocity, position, force, and torque. In the language of mathematics, physical vector quantities are represented by mathematical objects called **vectors**. We can add or subtract two vectors, and we can multiply a vector by a scalar or by another vector, but we cannot divide by a vector. The operation of division by a vector is not defined.





Source: University Physics Volume 1, OpenStax CNX

<https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/2-1-scalars-and-vectors>

#### Key Takeaways

*Basically:* a scalar has only magnitude, whereas a vector has magnitude and direction.

*Application:* I might have gone for a 4 km walk (scalar) but whether I walked in a straight line, took turns, or went 2 km out and turned around to walk 2 km back would tell me a lot more information (vector).

*Looking ahead:* We will talk about this again in sections 1.3 on vectors and in section 1.4 and 1.5 on dot products and cross products.

## 1.1.2 Newton's Laws

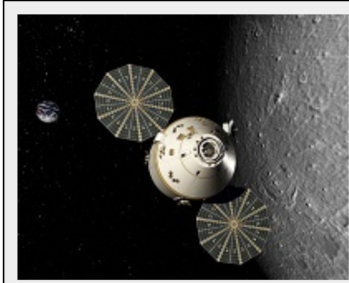
### 1st Law:

Newton's first law states that: **"A body at rest will remain at rest unless acted on by an unbalanced force. A body in motion continues in motion with the same speed and in the same direction unless acted upon by an unbalanced force."**

This law, also sometimes called the "law of inertia", means that bodies maintain their current velocity unless a force is applied to change that velocity. If an object is at rest with zero velocity it will remain at rest until some force begins to change that velocity, and if an object is moving at a set speed and in a set direction it will remain at that same velocity until some force begins to change that velocity.

**Net Forces:** It is important to note that the **net force** is what will cause a change in velocity. The net force is the sum of all forces acting on the body. For example, we can imagine gently pushing on the rock in the figure above and observing that the rock does not move. This is because we will have a friction force equal in magnitude and opposite in direction opposing our gentle pushing force. The sum of these two forces will be equal to zero, therefore the net force is zero and the change in velocity is zero.

**Rotational Motion:** Newton's first law also applies to moments and rotational velocities. A body will maintain its current rotational velocity until a net moment is exerted to change that rotational velocity. This can be seen in things like toy tops, flywheels, stationary bikes, and other objects that will continue spinning once started until brakes or friction stop them.



In the absence of friction in space, this space capsule will maintain its current velocity until some outside force causes that velocity to change. Public Domain image by NASA.



This rock is at rest with zero velocity and will remain at rest until a net force causes the rock to move. The net force on the rock is the sum of any force pushing the rock and the friction force of the ground on the rock opposing that force. Image by Liz Gray CC-BY-SA 2.0.



In the absence of friction, this spinning top would continue to spin forever, but the small frictional moment exerted at the point of contact between the top and the ground will slow the top's spinning over time. Image by Carrotmadman6 CC-BY-2.0.

Source: Engineering Mechanics, Jacob Moore et al.,  
[http://mechanicsmap.psu.edu/websites/1\\_mechanics\\_basics/newtons\\_first\\_law/firstlaw.html](http://mechanicsmap.psu.edu/websites/1_mechanics_basics/newtons_first_law/firstlaw.html)

## 2nd Law:

Newton's second law states that: **"When a net force acts on any body with mass, it produces an acceleration of that body. The net force will be equal to the mass of the body times the acceleration of the body"**

You will notice that the force and the acceleration in the equation above have an arrow above them. This means that they are vector quantities, having both a magnitude and a direction. Mass on the other hand is a scalar quantity having only a magnitude. Based on the above equation, you can infer that the magnitude of the net force acting on the body will be equal to the mass of the body times the magnitude of the acceleration, and that the direction of the net force on the body will be equal to the direction of the acceleration of the body.

**Rotational Motion:** Newton's second law also applies to moments and rotational velocities. The revised version of the second law equation states that the net moment acting on the object will be equal to the mass moment of inertia of the body about the axis of rotation ( $I$ ) times the angular acceleration of the body.

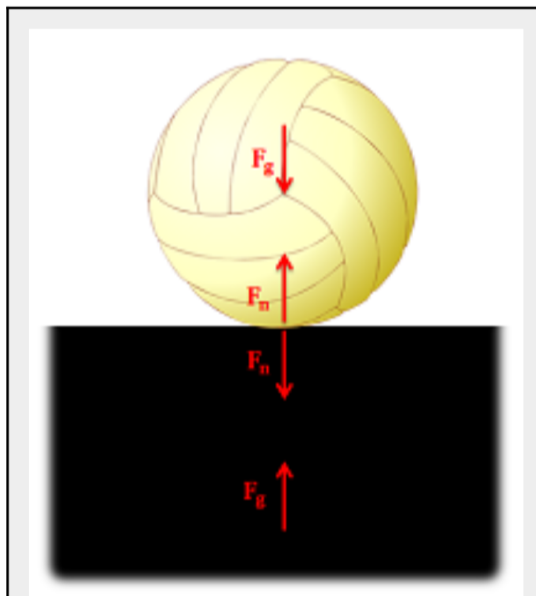
You should again notice that the moment and the angular acceleration of the body have arrows above them, indicating that they are vector quantities with both a magnitude and direction. The mass moment of inertia on the other hand is a scalar quantity having only a magnitude. The magnitude of the net moment will be equal to the mass moment of inertia times the magnitude of the angular acceleration, and the direction of the net moment will be equal to the direction of the angular acceleration.

Source: Engineering Mechanics, Jacob Moore et al., [http://mechanicsmap.psu.edu/websites/1\\_mechanics\\_basics/newtons\\_second\\_law/secondlaw.html](http://mechanicsmap.psu.edu/websites/1_mechanics_basics/newtons_second_law/secondlaw.html)

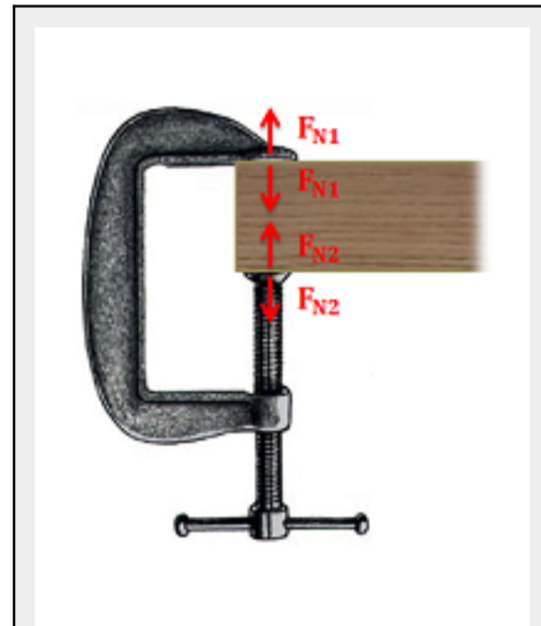
### 3rd Law:

Newton's Third Law states **"For any action, there is an equal and opposite reaction."** By "action" Newton meant a force, so for every force one body exerts on another body, that second body exerts a force of equal magnitude but opposite direction back on the first body. Since all forces are exerted by bodies (either directly or indirectly), all forces come in pairs, one acting on each of the bodies interacting.

Though there may be two equal and opposite forces acting on a single body, it is important to remember that for each of the forces a Third Law pair acts on a separate body. This can sometimes be confusing when there are multiple Third Law pairs at work. Below are some examples of situations where multiple Third Law pairs occur.



This volleyball resting on a surface has two pairs of Third Law forces. The first consists of the gravitational forces (one force on the ball and one force on the ground). The second consists of the normal forces at the point of contact (one force on the ball and one force on the ground). Image adapted from Public Domain image, no author listed.



If we ignore the weight of the two objects, this clamp will also have two pairs of Third Law forces. The first will be a set of normal forces at the top point of contact (one force on the wood and one force on the clamp) and the second will be another set of normal forces at the bottom point of contact (one force on the wood and one force on the clamp). Image adapted from Public Domain image, no author listed.

Source: Engineering Mechanics, Jacob Moore et al.,

[http://mechanicsmap.psu.edu/websites/1\\_mechanics\\_basics/newtons\\_third\\_law/thirdlaw.html](http://mechanicsmap.psu.edu/websites/1_mechanics_basics/newtons_third_law/thirdlaw.html)

### Key Takeaways

*Basically:* These 3 laws form the foundation of statics and dynamics. It makes our problems interesting! In statics, we don't do a lot with rotation.

- 1st law: the motion of an object won't change unless there is a force to cause the change.
- 2nd law: Combination of all forces = mass \* acceleration
- 3rd law: A system of interacting objects can be split up into parts, where forces are used to model the other part. Forces are equal (same size) and opposite (their directions cancel out – one up, one down)

*Application:* 1st law: a rock rolling down the hill will keep going unless it hits a tree. 2nd law: the amount of forces on the rock and how massive (heavy) it is will determine how much it is accelerating (or decelerating). 3rd law: the rock is pushing on the ground with the same amount of force as the ground is pushing on the rock, but in the opposite direction.

*Looking ahead:* You'll see these concepts again in Ch 7 on Inertia (1st law), Section 2.3 and 4.3 on equilibrium equations (2nd law), Section 4.2 on system free-body diagrams (3rd law).

## 1.1.3 Units

Giving numerical values for physical quantities and equations for physical principles allows us to understand nature much more deeply than qualitative descriptions alone. To comprehend these vast ranges, we must also have accepted units in which to express them. We shall find that even in the potentially mundane discussion of meters, kilograms, and seconds, a profound simplicity of nature appears: all physical quantities can be expressed as combinations of only seven base physical quantities.

We define a **physical quantity** either by specifying how it is measured or by stating how it is calculated from other measurements. For example, we might define distance and time by specifying methods for measuring them, such as using a meter stick and a stopwatch. Then, we could define *average speed* by stating that it is calculated as the total distance traveled divided by time of travel.

Measurements of physical quantities are expressed in terms of **units**, which are standardized values. For example, the length of a race, which is a physical quantity, can be expressed in units of meters (for sprinters) or kilometers (for distance runners). Without standardized units, it would be extremely difficult for scientists to express and compare measured values in a meaningful way.

Two major systems of units are used in the world: **SI units** (for the French *Système International d'Unités*), also

known as the *metric system*, and **English units** (also known as the *customary* or *imperial system*). English units were historically used in nations once ruled by the British Empire and are still widely used in the United States. English units may also be referred to as the *foot-pound-second* (fps) system, as opposed to the *centimeter-gram-second* (cgs) system.

### SI Units: Base and Derived Units

In any system of units, the units for some physical quantities must be defined through a measurement process. These are called the base quantities for that system and their units are the system's **base units**. All other physical quantities can then be expressed as algebraic combinations of the base quantities. Each of these physical quantities is then known as a **derived quantity** and each unit is called a **derived unit**. The choice of base quantities is somewhat arbitrary, as long as they are independent of each other and all other quantities can be derived from them. Typically, the goal is to choose physical quantities that can be measured accurately to a high precision as the base quantities. The reason for this is simple. Since the **derived units** can be expressed as algebraic combinations of the base units, they can only be as accurate and precise as the base units from which they are derived.

Based on such considerations, the International Standards Organization recommends using seven base quantities, which form the International System of Quantities (ISQ). These are the base quantities used to define the SI base units. The following table lists these seven ISQ base quantities and the corresponding SI base units.

ISQ Base Quantity	SI Base Unit
Length	Meter (m)
Mass	Kilogram (kg)
Time	Second (s)
Electrical current	Ampere (A)
Thermodynamic temp.	Kelvin (K)
Amount of substance	Mole (mol)
Luminous intensity	Candela (cd)

You are probably already familiar with some derived quantities that can be formed from the base quantities. For example, the geometric concept of area is always calculated as the product of two lengths. Thus, area is a derived quantity that can be expressed in terms of SI base units using square meters ( $m \times m = m^2$  ( $m \times m = m^2$ ))." role="presentation" style="font-family: proxima-nova, sans-serif;padding: 1px 0px;margin: 0px;font-size: 17.44px;vertical-align: baseline;background: #ffffff;border: 0px;line-height: 0;text-indent: 0px;text-align: left;text-transform: none;font-style: normal;font-weight: 400;letter-spacing: normal;float: none;direction: ltr;max-width: none;max-height: none;min-width: 0px;min-height: 0px;color: #373d3f">).

Similarly, volume is a derived quantity that can be expressed in cubic meters ( $m^3$ ). Speed is length per time; so in terms of SI base units, we could measure it in meters per second (m/s). Volume mass density (or just density) is mass per volume, which is expressed in terms of SI base units such as kilograms per cubic meter ( $kg/m^3$ ). Angles can also be thought of as derived quantities because they can be defined as the ratio of the arc length subtended by two radii of a circle to the radius of the circle. This is how the radian is defined. Depending on your background and interests, you may be able to come up with other derived quantities, such as the mass flow rate (kg/s) or volume flow rate ( $m^3/s$ ) of a fluid, electric charge (A·s), mass flux density [ $kg/(m^2 \cdot s)$ ], [ $kg/(m^2 \cdot s)$ ]," role="presentation" style="font-family: proxima-nova, sans-serif;padding: 1px 0px;margin: 0px;font-size: 17.44px;vertical-align: baseline;background: #ffffff;border: 0px;line-height: 0;text-indent: 0px;text-align: left;text-transform: none;font-style: normal;font-weight: 400;letter-spacing: normal;float: none;direction: ltr;max-width: none;max-height: none;min-width: 0px;min-height: 0px;color: #373d3f">).

none;min-width: 0px;min-height: 0px;color: #373d3f"> and so on. We will see many more examples throughout this text. For now, the point is that every physical quantity can be derived from the seven base quantities, and the units of every physical quantity can be derived from the seven SI base units.

Source: University Physics Volume 1, OpenStax CNX,  
<https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/1-2-units-and-standards/>

While most Canadian companies use SI, much manufacturing still uses English units, so it's important for you to be familiar with them. What is a big number in feet? What is small? It's important to know. The most important advice is to **stay in one unit system**. So if you are doing a homework problem that has a mixture, convert to one system to be consistent. Challenge your self to try the one you aren't comfortable with. Here is a table of the most common quantities that we'll use in this class:

Quantity	SI Unit	English
Length	m (meter), km (kilometer), mm (millimeter)	ft (foot), mi (mile), in (inch)
Mass	kg (kilogram)	slug
Force	N (Newton)	lb (pound)
Pressure	Pa (Pascal) = $\text{N/m}^2$	psi (pound per square inch) = $\text{lb/in}^2$

Very helpful additional information about units is at this webpage:

<https://www.physics.nist.gov/cuu/Units/index.html>

#### Key Takeaways

*Basically:* Units give us a standard so we can use the same language to describe a concept.

*Application:* In 1999, after taking 286 days for NASA Mars Orbiter satellite to get to Mars, a conversion error between N and lb caused the \$125 million satellite to be lost, forever. Click [here](#) for more fun conversion error stories. If you want to design ANYTHING, you need to be sure everyone involved is using the same unit system.

*Looking Ahead:* The next section (1.1.4) will look at converting the units back and forth between the two systems.

## 1.1.4 Measurement Conversions

It is often necessary to convert from one unit to another. For example, if you are reading a European cookbook, some quantities may be expressed in units of liters and you need to convert them to cups. Or perhaps you are reading walking directions from one location to another and you are interested in how many miles you will be walking. In this case, you may need to convert units of feet or meters to miles.

Let's consider a simple example of how to convert units. Suppose we want to convert 80 m to kilometers. The first thing to do is to list the units you have and the units to which you want to convert. In this case, we have units in *meters* and we want to convert to *kilometers*. Next, we need to determine a conversion factor relating meters to kilometers. A **conversion factor** is a ratio that expresses how many of one unit are equal to another unit. For example, there are 12 in. in 1 ft, 1609 m in 1 mi, 100 cm in 1 m, 60 s in 1 min, and so on. In this case, we know that there are 1000 m in 1 km. Now we can set up our unit conversion. We write the units we have and then multiply them by the conversion factor so the units cancel out, as shown:

$$80 \text{ m} \times \frac{1 \text{ km}}{1000 \text{ m}} = 0.080 \text{ km}$$

Note that the unwanted meter unit cancels, leaving only the desired kilometer unit. You can use this method to convert between any type of unit. Now, the conversion of 80 m to kilometers is simply the use of a metric prefix, as we saw in the preceding section, so we can get the same answer just as easily by noting that

$$80 \text{ m} = 8.0 \times 10^1 \text{ m} = 8.0 \times 10^{-2} \text{ km} = 0.080 \text{ km}$$

$$80 \text{ m} = 8.0 \times 10^1 \text{ m} = 8.0 \times 10^{-2} \text{ km} = 0.080 \text{ km}$$

since “kilo-” means  $10^3$  and  $1 = -2 + 3$ . However, using conversion factors is handy when converting between units that are not metric or when converting between derived units, as the following examples illustrate.

Source: University Physics Volume 1, OpenStax CNX,  
<https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/1-3-unit-conversion/>

Going back and forth between SI and English will become very useful skill. If you can memorize km to mi, ft to m, and inches to ft, you'll be able to communicate better with coworkers. Here are common conversions you'll need for this course:

Quantity	SI	English	Convert
Length	1 km = 1000 m 1 m = 1000 mm	1 mi = 5,280 ft 1 ft = 12 in	1 m = 3.28 ft 2.2 km = 1 mi
Mass	kg	slug	1 slug = 14.6 kg
Force	N	lb	1 lb = 4.448 N
Pressure	Pa	psi	1psi = 6895 Pa

All of the other units that we will encounter will be a mix of these units (intensity  $w = \text{N/m}$  or  $\text{lb/ft}$ ). One additional conversion that is common is  $1 \text{ lb} = 2.2 \text{ kg}$ , though this only works on Earth because it is mixing kg and lb (see next section). For a full table, MechanicsMap has a pdf available: <http://mechanicsmap.psu.edu/websites/UnitConversion.pdf>



## Units outside the SI that are accepted for use with the SI

Name	Symbol	Value in SI units
minute (time)	min	1 min = 60 s
hour	h	1 h = 60 min = 3600 s
day	d	1 d = 24 h = 86 400 s
degree (angle)	°	1° = ( $\pi$ /180) rad
minute (angle)	'	1' = (1/60)° = ( $\pi$ /10 800) rad
second (angle)	"	1" = (1/60)' = ( $\pi$ /648 000) rad
liter	L	1 L = 1 dm <sup>3</sup> = 10 <sup>-3</sup> m <sup>3</sup>
metric ton <sup>(a)</sup>	t	1 t = 10 <sup>3</sup> kg
neper	Np	1 Np = 1
bel <sup>(b)</sup>	B	1 B = (1/2) ln 10 Np <sup>(c)</sup>
electronvolt <sup>(d)</sup>	eV	1 eV = 1.602 18 x 10 <sup>-19</sup> J, approximately
unified atomic mass unit <sup>(e)</sup>	u	1 u = 1.660 54 x 10 <sup>-27</sup> kg, approximately
astronomical unit <sup>(f)</sup>	au	1 au = 149 597 870 700 m, exactly

<sup>(a)</sup> In many countries, this unit is called "tonne."

<sup>(b)</sup> The bel is most commonly used with the SI prefix deci: 1 dB = 0.1 B.

<sup>(c)</sup> Although the neper is coherent with SI units and is accepted by the CIPM, it has not been adopted by the General Conference on Weights and Measures (CGPM, *Conférence Générale des Poids et Mesures*) and is thus not an SI unit.

<sup>(d)</sup> The electronvolt is the kinetic energy acquired by an electron passing through a potential difference of 1 V in vacuum. The value must be obtained by experiment, and is therefore not known exactly.

<sup>(e)</sup> The unified atomic mass unit is equal to 1/12 of the mass of an unbound atom of the nuclide <sup>12</sup>C, at rest and in its ground state. The value must be obtained by experiment, and is therefore not known exactly.

<sup>(f)</sup> The astronomical unit of length was redefined by the XXVIII General Assembly of the International Astronomical Union (Resolution B2, 2012).

Source: <https://www.physics.nist.gov/cuu/Units/outside.html>

## Key Takeaways

*Basically:* Different industries use different standards. English is common in the US. SI is standard many other places, however not generally in aerospace.

*Application:* In 1999, after taking 286 days for NASA Mars Orbiter satellite to get to Mars, a conversion error between N and lb caused the \$125 million satellite to be lost, forever. Click [here](#) for more fun conversion error stories. If you want to design ANYTHING, you need to be sure everyone involved is using the same unit system.

*Looking Ahead:* Always always always check what unit you're using. So many students lose points on homework and the test because they aren't paying attention to units.

## 1.1.5 Weight vs. Mass

Weight is the force exerted by gravity. While all objects with mass exert an attractive force of gravity on all other objects with mass, that force is usually negligible unless the mass of one of the objects is very large. For an object near the surface of the Earth, we can, to a very good degree of approximation, assume that the only force of gravity on the object is from the Earth. We usually label the force of gravity on an object as  $F_g$ . All objects near the surface of the Earth will experience a weight, as long as they have a mass. If an object has a mass,  $m$ , and is located near the surface of the Earth, it will experience a force (its weight) that is given by:

$$\vec{F}_g = m\vec{g}$$

where  $\vec{g}$  is the Earth's "gravitational field" vector and points towards the centre of the Earth. Near the surface of the Earth, the magnitude of the gravitational field is approximately  $g = 9.81 \text{ m/s}^2$ . The gravitational field is a measure of the strength of the force of gravity from the Earth (it is the gravitational force per unit mass). The magnitude of the gravitational field is weaker as you move further from the centre of the Earth (e.g. at the top of a mountain, or in Earth's orbit). The gravitational field is also different on different planets; for example, at the surface of the moon, it is approximately  $g_m = 1.62 \text{ m/s}^2$  (six times less) - thus the weight of an object is six times less at the surface of the moon (but its mass is still the same). As we will see, the magnitude of the gravitational field from any spherical body of mass  $M$  (e.g a planet) is given by:

$$g(r) = G \frac{M}{r^2}$$

where  $G = 6.67 \times 10^{-11}$  is Newton's constant of gravity, and  $r$  is the distance from the centre of the object.

Although we have not yet introduced the concept of mass, it is worth emphasizing that mass and weight are different (they have different dimensions). Mass is an intrinsic property of an object, whereas weight is a force of gravity that is exerted on that object because it has mass and is located next to another object with mass (e.g. the Earth). On Earth, when we measure our weight, we usually do so by standing on a spring scale, which is designed to measure a force by compressing a spring. We are thus measuring  $mg$ , which can easily be related to our mass since, on Earth, weight and mass are related by a factor of  $g = 9.81 \text{ m/s}^2$ ; this is usually what leads to the confusion between mass and weight.

Source: Introductory Physics, Ryan Martin et al.,  
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=4c3c2c75-0029-4c9e-967f-41f178bebbbbb> page 106

In the English language, the words ‘mass’ and ‘weight’ are used interchangeably. A person might say, “I weigh 50 kg”, but in statics language, that’s wrong! Or more accurately, that language isn’t precise enough for statics.



An object’s mass is the same whether they are on the moon, Mars, or Earth. However, their weight changes because the constant of gravity with which that planet is pulling changes (see above description comparing the moon and Earth).

$$g = 9.81 \text{ m/s}^2 \text{ (SI) and } g = 32.2 \text{ ft/s}^2 \text{ (English)}$$

$$\vec{F}_g = m \vec{g}$$

Weight = mass \* gravitational constant

Units of mass are kg (SI) or slugs (English) whereas units of weight/force are N (SI) or lb (English). Because ‘slugs’ is such an odd, unfamiliar unit, the graphic on the left uses real slugs to help you remember to say “my mass is 50 kg (or 3.43 slug)” or “I weigh 490 N (or 110 lb)”.

While most Canadian companies use SI units, it’s important to be familiar with English, so you should learn slugs. You don’t want to be excluded from a conversation at your future job.

Note  $\text{lb}_m$  (pound-mass) is not used in this book, though some textbooks use it as a mass value. When lb is used, it is assumed to be  $\text{lb}_f$  (pound-force).

#### Key Takeaways

*Basically:* Mass and force are two different quantities. Mass is in kg (SI) or slug (English) and weight is in N (SI) and lb (English).

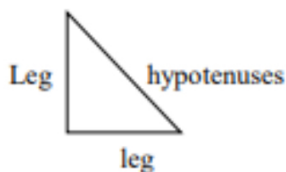
*Application:* Mass stays the same, but weight changes from the Earth to the moon.

*Looking ahead:* This will become very important when we look at forces in Section 4.1.

## 1.1.6 Pythagorean Theorem

**Right triangle:** a triangle containing a  $90^\circ$  angle.

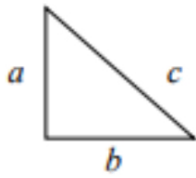
**Pythagorean theorem:** a relation among the three sides of a right triangle which states that the square of the hypotenuse is equal to the sum of the squares of the other two sides (legs).



$$\text{hypotenuse}^2 = \text{leg}^2 + \text{leg}^2$$

$$\text{hypotenuse} = \sqrt{\text{leg}^2 + \text{leg}^2}, \quad \text{leg} = \sqrt{\text{hypotenuse}^2 - \text{leg}^2}$$

Using the Pythagorean theorem can find the length of the missing side in a right triangle.



$$c = \sqrt{a^2 + b^2}$$

$$b = \sqrt{c^2 - a^2}$$

$$a = \sqrt{c^2 - b^2}$$

- $c$  is the longest side of the triangle (hypotenuses).
- Other two sides (legs) of the triangle  $a$  and  $b$  can be exchanged.

Source: Key Concepts of Intermediate Level Math, Meizhong Wang and the College of New Caledonia,  
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=d8bdc88b-5439-4652-b4bb-2948f0d5c625>, page 136.

A special cases of the right triangle is called a 3-4-5 triangle, or a Pythagorean triple. The two short sides are 3 and 4, and the hypotenuse is 5! Many of your homework problems will use this coincidence so you can save on the math by remembering 3-4-5 triangles:  $3^2 + 4^2 = 5^2$ ,  $9 + 16 = 25$ . Wow!

#### Key Takeaways

*Basically:* The pythagorean theorem will help you find a lot of information throughout this course. The longest side  $c^2 = a^2 + b^2$

*Application:* If I have a 6 ft ladder leaned up against a wall whose base is 2 ft from the wall, the pythagorean theorem helps you to calculate the vertical height of the ladder ( $b^2 = 6^2 - 2^2$ ).

*Looking Ahead:* You'll use this to help find geometrical aspects of the problems, especially when we get into trusses in Ch 5.

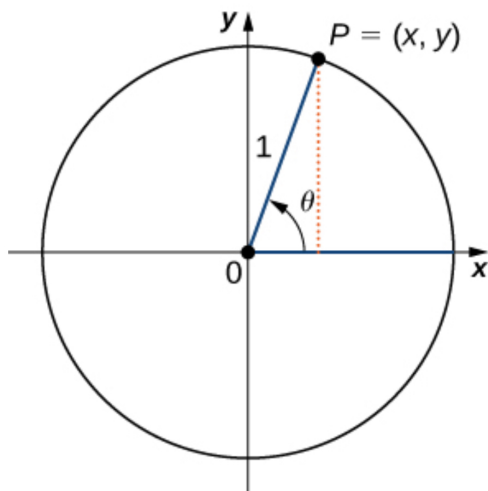
## 1.1.7 Sine/Cosine Law's

### The Six Basic Trigonometric Functions

Trigonometric functions allow us to use angle measures, in radians or degrees, to find the coordinates of a point on any circle—not only on a unit circle—or to find an angle given a point on a circle. They also define the

relationship among the sides and angles of a triangle.

To define the trigonometric functions, first consider the unit circle centered at the origin and a point  $P=(x,y)$  on the unit circle. Let  $\theta$  be an angle with an initial side that lies along the positive  $x$ -axis and with a terminal side that is the line segment  $OP$ . We can then define the values of the six trigonometric functions for  $\theta$  in terms of the coordinates  $x$  and  $y$  of the point  $P$ .



Let  $P=(x,y)$  be a point on the unit circle centered at the origin  $O$ . Let  $\theta$  be an angle with an initial side along the positive  $x$ -axis and a terminal side given by the line segment  $OP$ . The trigonometric functions are then defined as

$$\sin \theta = y, \quad \csc \theta = \frac{1}{y}, \quad \cos \theta = x, \quad \sec \theta = \frac{1}{x}, \quad \tan \theta = \frac{y}{x}, \quad \cot \theta = \frac{x}{y}$$

$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$

ot\theta=\frac{x}{y}\$\$

If  $x=0$ ,  $\sec \theta$  and  $\tan \theta$  are undefined. If  $y=0$ , then  $\cot \theta$  and  $\csc \theta$  are undefined.

We can see that for a point  $P=(x,y)$  on a circle of radius  $r$  with a corresponding angle  $\theta$ , the coordinates  $x$  and  $y$  satisfy:

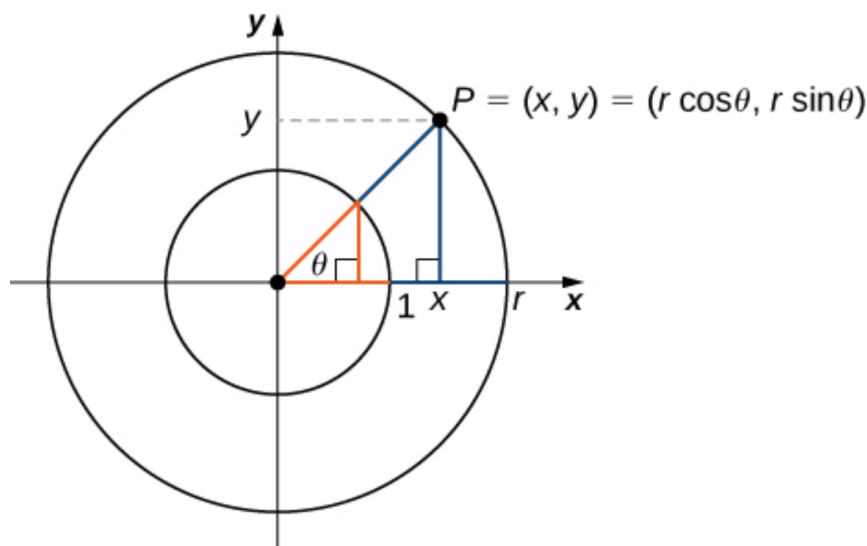
$$\cos \theta = \frac{x}{r}$$

$$x = r \cos \theta$$

$$\sin \theta = \frac{y}{r}$$

$$y = r \sin \theta$$

The values of the other trigonometric functions can be expressed in terms of  $x$ ,  $y$ , and  $r$ :



The table below shows the values of sine and cosine at the major angles in the first quadrant. From this table, we can determine the values of sine and cosine at the corresponding angles in the other quadrants. The values of the other trigonometric functions are calculated easily from the values of  $\sin \theta$  and  $\cos \theta$ :

$\theta$	$\sin \theta$	$\cos \theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

### Trigonometric Identities

A trigonometric identity is an equation involving trigonometric functions that is true for all angles  $\theta$ . A trigonometric identity is an equation involving trigonometric functions that is true for all angles  $\theta$  for which the functions are defined. We can use the identities to help us solve or simplify equations. The main trigonometric identities are listed next.

#### RULE: TRIGONOMETRIC IDENTITIES

##### Reciprocal identities

$$\begin{aligned}\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta} \\ \csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta}\end{aligned}$$

##### Pythagorean identities

$$\sin^2 \theta + \cos^2 \theta = 1 \quad 1 + \tan^2 \theta = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta$$

##### Addition and subtraction formulas

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

##### Double-angle formulas

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

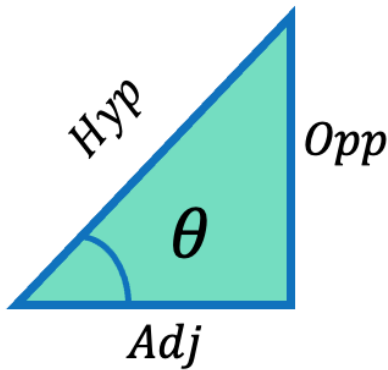
$$\cos(2\theta) = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta = \cos^2 \theta - \sin^2 \theta$$

Source: Calculus Volume 1, Gilbert Strang & Edwin “Jed” Herman,  
<https://openstax.org/books/calculus-volume-1/pages/1-3-trigonometric-functions>



We often refer to this as SOH-CAH-TOA:

- Sine = Opposite / Hypotenuse >>  $S = O/H$  >> SOH
- Cosine = Adjacent / Hypotenuse >>  $C = A/H$  >> CAH
- Tangent = Opposite / Adjacent >>  $T = O/A$  >> TOA



$$\text{SOH: } \sin \theta = \text{Opp} / \text{Hyp}$$

$$\text{CAH: } \cos \theta = \text{Adj} / \text{Hyp}$$

$$\text{TOA: } \tan \theta = \text{Opp} / \text{Adj}$$

I remember that *cos* is *close* - the side that's close to the angle is cosine. (It kind of rhymes and 'close' is a more familiar word than 'adjacent').

#### Key Takeaways

*Basically:* Trigonometric functions will help you to solve problems. You'll use SOH-CAH-TOA in many statics problems, whether to componentize a vector or resolve a force.

*Application:* A 6ft ladder leaning up against a house is at a 60 degree angle. We can find the vertical height where the ladder reaches the house by using height = 6 ft \* sin 60 degrees. (sin = opp / hyp)

*Looking Ahead:* Chapter 4 (forces) and Chapter 5 (trusses) will use calculation of angles a lot.

# 1.2 XYZ Coordinate Frame

We need a standard to be able to share a common language. The *Cartesian coordinate frame* lets us express the location of a point so that others can understand what we're talking about. In this section, we'll look at 2d and 3d coordinate frames.

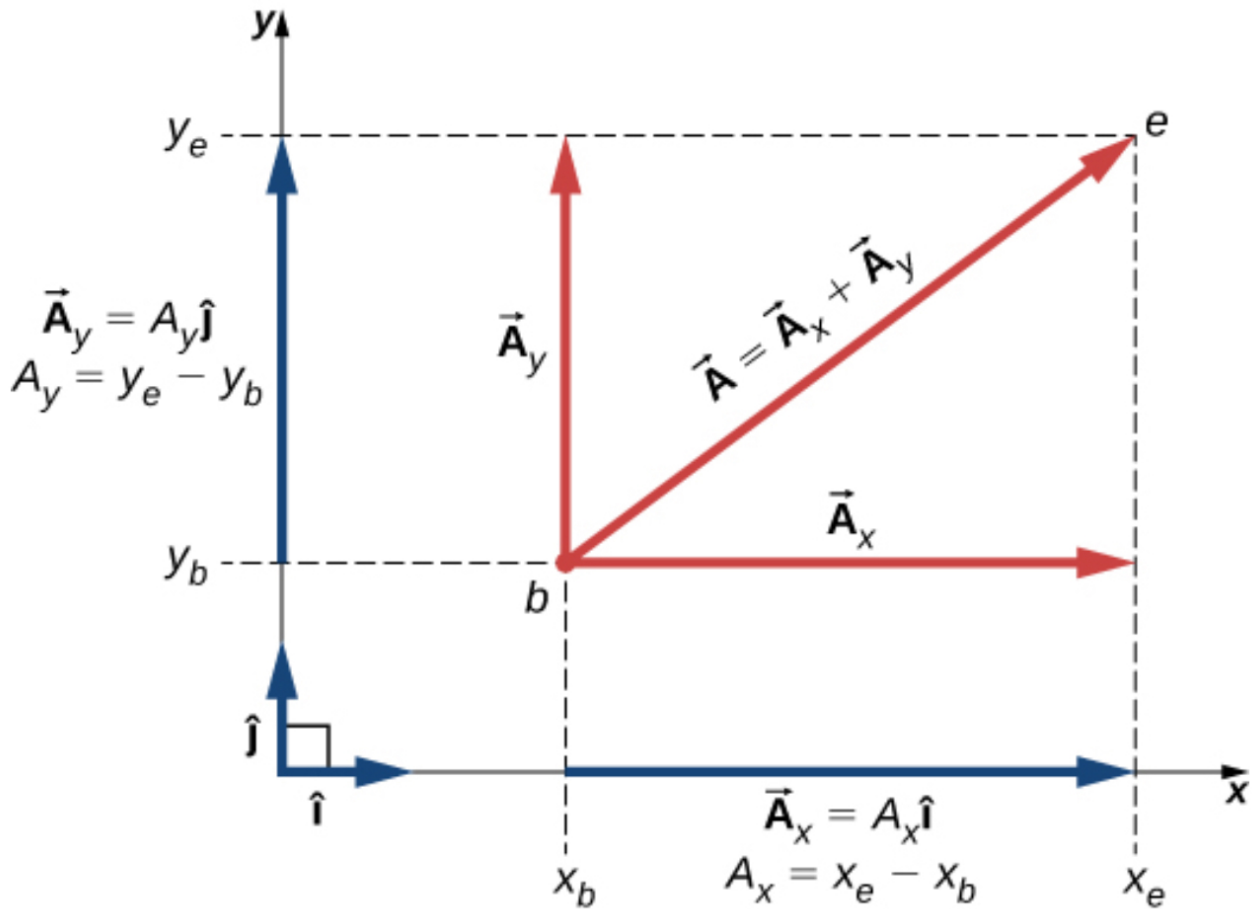
## 1.2.1 Cartesian Coordinate Frame in 2D

Vectors are usually described in terms of their components in a **coordinate system**. Even in everyday life we naturally invoke the concept of orthogonal projections in a rectangular coordinate system. For example, if you ask someone for directions to a particular location, you will more likely be told to go 40 km east and 30 km north than 50 km in the direction  $37^\circ$  north of east.

In a rectangular (Cartesian) x-y coordinate system in a plane, a point in a plane is described by a pair of coordinates  $(x, y)$ . In a similar fashion, a vector  $\vec{A}$  in a plane is described by a pair of its *vector* coordinates. The x-coordinate of vector  $\vec{A}$  is called its x-component and the y-coordinate of vector  $\vec{A}$  is called its y-component. The vector x-component is a vector denoted by  $\vec{A}_x$ . The vector y-component is a vector denoted by  $\vec{A}_y$ . In the Cartesian system, the x and y **vector components** of a vector are the orthogonal projections of this vector onto the x- and y-axes, respectively. In this way, following the parallelogram rule for vector addition, each vector on a Cartesian plane can be expressed as the vector sum of its vector components:

$$\vec{A} = \vec{A}_x + \vec{A}_y$$

As illustrated in the figure below, vector  $\vec{A}$  is the diagonal of the rectangle where the x-component  $\vec{A}_x$  is the side parallel to the x-axis and the y-component  $\vec{A}_y$  is the side parallel to the y-axis. Vector component  $\vec{A}_x$  is orthogonal to vector component  $\vec{A}_y$ .



**Figure 2.16** Vector  $\vec{A}$  in a plane in the Cartesian coordinate system is the vector sum of its vector x- and y-components. The x-vector component  $\vec{A}_x$  is the orthogonal projection of vector  $\vec{A}$  onto the x-axis. The y-vector component  $\vec{A}_y$  is the orthogonal projection of vector  $\vec{A}$  onto the y-axis. The numbers  $A_x$  and  $A_y$  that multiply the unit vectors are the scalar components of the vector.

It is customary to denote the positive direction on the x-axis by the unit vector  $\hat{i}$  and the positive direction on the y-axis by the unit vector  $\hat{j}$ . **Unit vectors of the axes**,  $\hat{i}$  and  $\hat{j}$ , define two orthogonal directions in the plane. As shown in the figure above, the x- and y- components of a vector can now be written in terms of the unit vectors of the axes:

$$\vec{A}_x = A_x \hat{i}$$

$$\vec{A}_y = A_y \hat{j}$$

The vectors  $\vec{A}_x$  and  $\vec{A}_y$  defined by the figure above are the *vector components* of vector  $\vec{A}$ . The numbers  $A_x$  and  $A_y$  that define the vector components above are the *scalar components* of vector  $\vec{A}$ . The numbers  $A_x$  and  $A_y$  that define the vector components above are the scalar components of vector  $\vec{A}$ .

role="presentation" style="font-family: proxima-nova, sans-serif; padding: 1px 0px; margin: 0px; font-size: 17.44px; vertical-align: baseline; background: transparent; border: 0px; line-height: 0; text-indent: 0px; text-align: left; text-transform: none; font-style: normal; font-weight: normal; letter-spacing: normal; float: none; direction: ltr; max-width: none; max-height: none; min-width: 0px; min-height: 0px;">>A→" role="presentation" style="font-family: proxima-nova, sans-serif; padding: 1px 0px; margin: 0px; font-size: 17.44px; vertical-align: baseline; background: transparent; border: 0px; line-height: 0; text-indent: 0px; text-align: left; text-transform: none; font-style: normal; font-weight: normal; letter-spacing: normal; float: none; direction: ltr; max-width: none; max-height: none; min-

width: 0px; min-height: 0px;">> $\vec{A}$ . Combining the diagram above with the equations above, we obtain **the component form of a vector:**

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

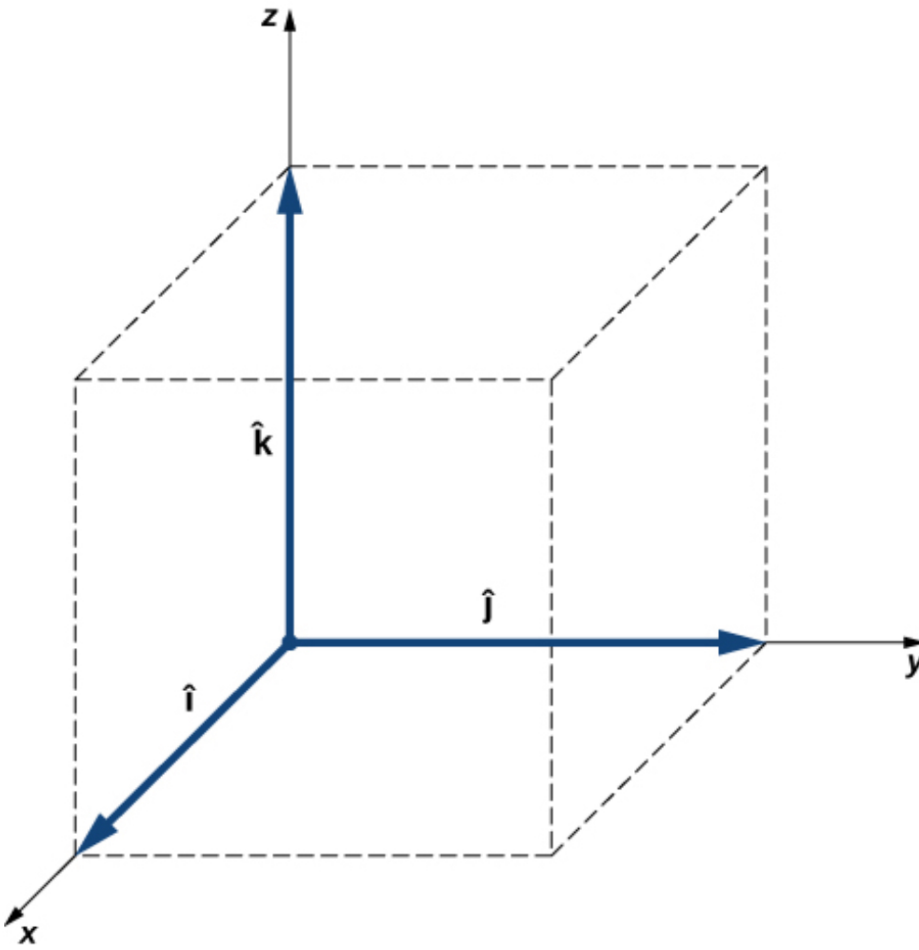
If we know the coordinates  $b(x_b, y_b)$  of the origin point of a vector (where  $b$  stands for “beginning”) and the coordinates  $e(x_e, y_e)$  of the end point of a vector (where  $e$  stands for “end”), we can obtain the scalar components of a vector simply by subtracting the origin point coordinates from the end point coordinates:

$$A_x = x_e - x_b$$

$$A_y = y_e - y_b$$

## 1.2.2. Cartesian Coordinate Frame in 3D

To specify the location of a point in space, we need three coordinates  $(x, y, z)$ , where coordinates  $x$  and  $y$  specify locations in a plane, and coordinate  $z$  gives a vertical position above or below the plane. Three-dimensional space has three orthogonal directions, so we need not two but *three* unit vectors to define a three-dimensional coordinate system. In the Cartesian coordinate system, the first two unit vectors are the unit vector of the  $x$ -axis  $\hat{i}$  and the unit vector of the  $y$ -axis  $\hat{j}$ . The third unit vector  $\hat{k}$  is the direction of the  $z$ -axis, as can be seen below. The order in which the axes are labeled, which is the order in which the three unit vectors appear, is important because it defines the orientation of the coordinate system. The order  $x$ - $y$ - $z$ , which is equivalent to the order  $\hat{i}$ - $\hat{j}$ - $\hat{k}$ , defines the standard right-handed coordinate system (positive orientation).



In three-dimensional space, vector  $\vec{A}$  has three vector components: the x-component  $\vec{A}_x = A_x \hat{i}$ , which is the part of vector  $\vec{A}$  along the x-axis; the y-component  $\vec{A}_y = A_y \hat{j}$ , which is the part of  $\vec{A}$  along the y-axis; and the z-component  $\vec{A}_z = A_z \hat{k}$ , which is the part of the vector along the z-axis. A vector in three-dimensional space is the vector sum of its three vector components:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

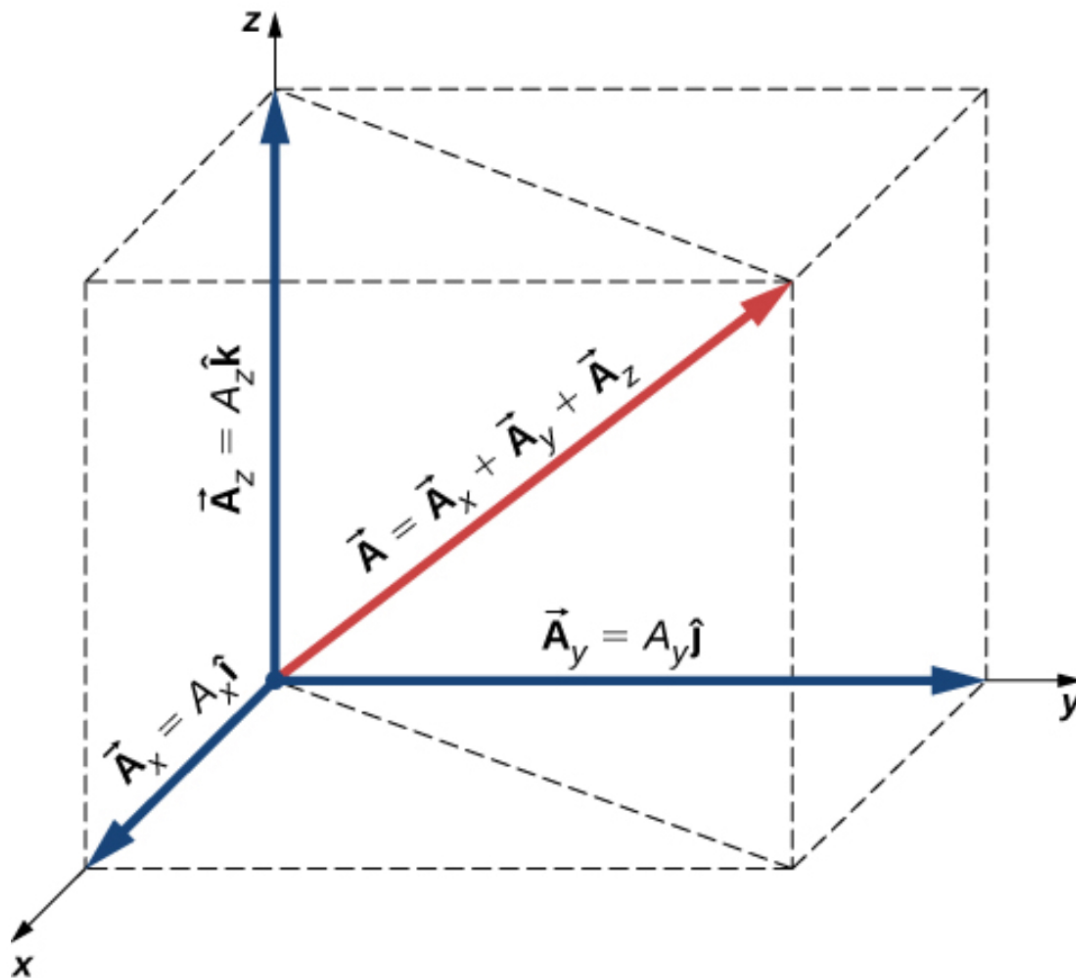
If we know the coordinates of its origin  $b(x_b, y_b, z_b)$  and of its end  $e(x_e, y_e, z_e)$  its scalar components are obtained by taking their differences, and the z-component is given by:

$$A_z = z_e - z_b$$

Magnitude  $A$  is obtained by the following equation:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

This expression for the vector magnitude comes from applying the Pythagorean theorem twice. As seen in the figure below, the diagonal in the  $x$ - $y$  plane has length  $\sqrt{A_x^2 + A_y^2}$  and its square adds to the square  $A_z^2$  to give  $A^2$ . Note that when the  $z$ -component is zero, the vector lies entirely in the  $x$ - $y$  plane and its description is reduced to two dimensions.



Source: University Physics Volume 1, OpenStax CNX,  
<https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/2-2-coordinate-systems-and-components-of-a-vector/>

The Cartesian coordinate frame is a right-orthogonal system. This will matter when we start looking at the right-hand rule in section 3.1. What it means is that when you draw two of the directions (say  $x$  and  $y$ ), then  $z$  must go either up or down. We'll get into that more, later.

### Key Takeaways

*Basically:* The Cartesian coordinate frame gives us a way to communicate the location of a point in space. In 2-d we use  $[x, y]$  and in 3-d we also include  $z$ :  $[x, y, z]$ .

*Application:* If I am trying to walk across the room, I can walk in a straight line for 5 steps, or I can take 3 steps to the right and 4 steps ahead. If I wanted to describe the position of where I went in the Cartesian coordinate frame, it would be  $[3, 4]$  assuming  $x$  is to the right and  $y$  is straight ahead.

*Looking ahead:* This will connect with everything we do in this class, especially with cross-products (1.5), torque (1.6), and equilibrium equations (everywhere).

# 1.3 Vectors

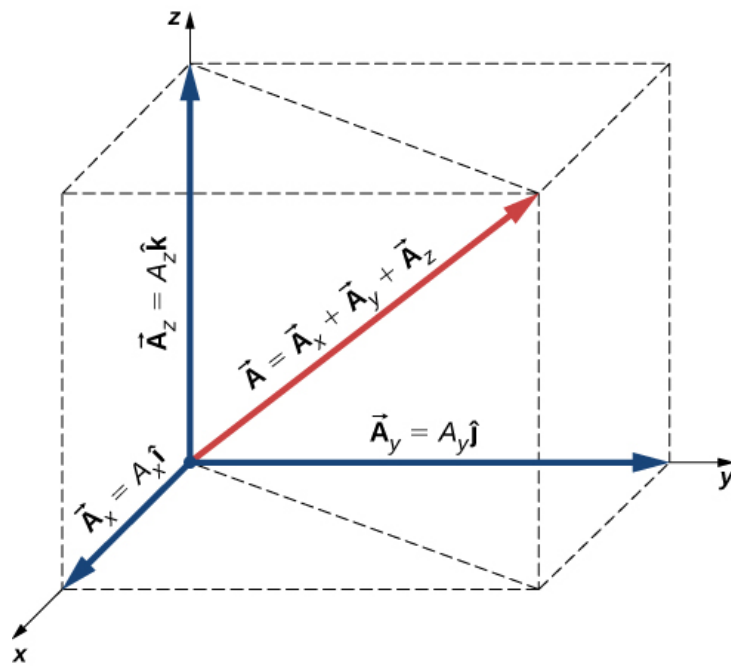
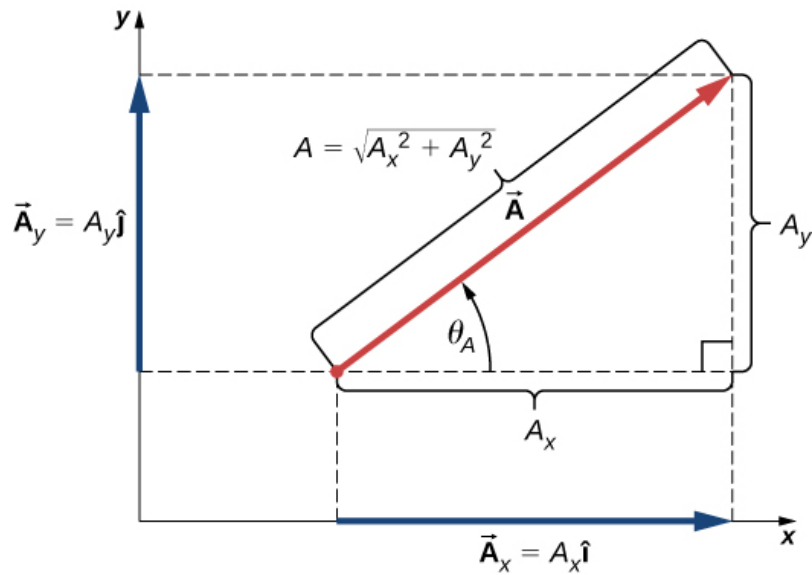
## 1.3.1 Vector Components

Some fun facts about vectors:

- The vector is denoted with a line on top or bottom:  $\vec{A}$  or  $\underline{A}$ .
- There are two parts of a vector ( $\vec{A}$ ): magnitude ( $A$  or  $|\underline{A}|$ ) and direction ( $\underline{\hat{a}}$ ):  $\vec{A} = |\underline{A}| \underline{\hat{a}}$
- In 2-dimensions, there are two *components*: x and y. In 3-d, there are three components: x, y, and z.
- Vectors can be denoted using Cartesian form or brackets:  $\vec{A} = A_x \underline{\hat{i}} + A_y \underline{\hat{j}} + A_z \underline{\hat{k}}$  or using the bracket form horizontally:  $\vec{A} = [A_x, A_y, A_z]$  or vertically:  $\vec{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$
- The magnitude ( $A$  or  $|\underline{A}|$ ) is calculated using the Pythagorean theorem for each component in 2d:  $A = \sqrt{A_x^2 + A_y^2}$  and 3d:  $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$
- The unit vector ( $\underline{\hat{u}}$ ) represents the direction in cartesian form  $\underline{\hat{u}} = \underline{\hat{i}} + \underline{\hat{j}} + \underline{\hat{k}}$  or using bracket form:  $[\underline{\hat{i}}, \underline{\hat{j}}, \underline{\hat{k}}]$ .
- The magnitude of the unit vector is 1 (denoted by the 'hat' on top) and it is unit-less:  $|\underline{\hat{u}}| = \sqrt{\underline{\hat{i}}^2 + \underline{\hat{j}}^2 + \underline{\hat{k}}^2} = 1$
- The unit vector can be calculated from the magnitude and vector:  $\underline{\hat{a}} = \vec{A} / A$

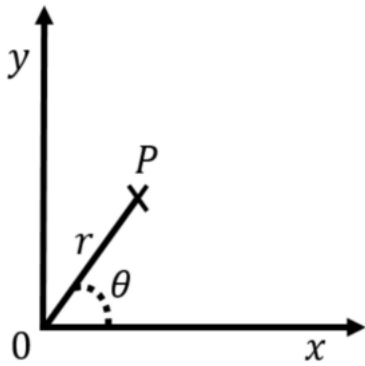
In 2d & 3d:





Source: University Physics Volume 1, OpenStax CNX,  
<https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/2-2-coordinate-systems-and-components-of-a-vector/>

## 1.3.2 Componentizing a Vector



Source: Introductory Physics,  
Ryan Martin et al.,  
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=4c3c2c75-0029-4c9e-967f-41f178bebbb>  
p814

In 2d:

To find the components of a vector ( $\underline{A}$ ) in 2 dimensions (the x and y portions  $A_x$  and  $A_y$ ), use SOH CAH TOA:

$$\underline{A} = A_x \underline{\hat{i}} + A_y \underline{\hat{j}}$$

$$A_x = |\underline{A}| \cos(\Theta)$$

$$A_y = |\underline{A}| \sin(\Theta)$$

$$|\underline{A}|^2 = A_x^2 + A_y^2 \text{ (magnitude)}$$

$$\tan(\Theta) = A_x / A_y \text{ (direction)}$$

In 3d:

$$\underline{A} = A_x \underline{\hat{i}} + A_y \underline{\hat{j}} + A_z \underline{\hat{k}}$$

$$|\underline{A}|^2 = A_x^2 + A_y^2 + A_z^2 \text{ (magnitude)}$$

$$\hat{a} = \frac{\underline{A}}{|\underline{A}|} \quad \hat{a}_x = \frac{A_x}{|\underline{A}|}$$

$$\sqrt{\left(\hat{\mathbf{i}}A_x + \hat{\mathbf{j}}A_y + \hat{\mathbf{k}}A_z\right)^2} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

### 1.3.3 Position Vector

The position vector describes the position of an object or person from a predefined origin (a starting point, absolute 0, or some other point), for example the point where  $\underline{\mathbf{a}}$  in the above image is point at.  $\underline{\mathbf{a}}$  is the position vector. You can add individual position vectors to find the total position traveled ( $\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$ ), for example if someone walks from one point on campus to another, they would rarely walk in one straight line like  $\underline{\mathbf{c}}$ . In the image below, imagine that there is a building in the square near where  $\underline{\mathbf{a}}$  and  $\underline{\mathbf{b}}$  meet, so the person couldn't take  $\underline{\mathbf{c}}$  but had to walk around. The total distance traveled is  $|\underline{\mathbf{a}}| + |\underline{\mathbf{b}}|$ , not  $|\underline{\mathbf{c}}|$  (because  $|\underline{\mathbf{c}}| \neq |\underline{\mathbf{a}}| + |\underline{\mathbf{b}}|$ ).

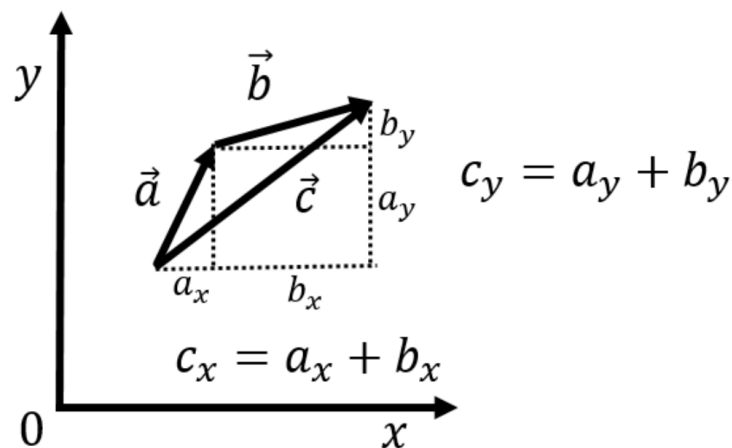


Figure A.10: Geometric addition of the vectors  $\vec{a}$  and  $\vec{b}$  by placing them “head to tail”.

Source: Introductory Physics, Ryan Martin et al.,  
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=4c3c2c75-0029-4c9e-967f-41f178bebbbb> page 821

Subtraction works the same way, but instead of going from tail to head of the arrow, the reverse direction is taken, from head to tail. For example,  $\underline{\mathbf{a}} = \underline{\mathbf{c}} - \underline{\mathbf{b}}$ , follow  $\underline{\mathbf{c}}$  from tail to head, then go in the reverse direction of  $\underline{\mathbf{b}}$  from head to tail, and you end up at  $\underline{\mathbf{a}}$ .

## 1.3.4 Vector Math

Here's more official language to describe vectors:

Vectors can be added together and multiplied by scalars. Vector addition is associative and commutative, and vector multiplication by a sum of scalars is distributive. Also, scalar multiplication by a sum of vectors is distributive:

$$[\text{latex}]\alpha(\vec{A} + \vec{B}) = \alpha\vec{A} + \alpha\vec{B}[/\text{latex}]$$

In this equation,  $\alpha$  is any number (a scalar). For example, a vector antiparallel to vector  $[\text{latex}]\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}[/\text{latex}]$  can be expressed simply by multiplying  $[\text{latex}]\vec{A}[/\text{latex}]$  by the scalar  $\alpha = -1$ :

$$[\text{latex}]-\vec{A} = -A_x\hat{i} - A_y\hat{j} - A_z\hat{k}[/\text{latex}]$$

The generalization of the number zero to vector algebra is called the null vector, denoted by  $[\text{latex}]\vec{0}[/\text{latex}]$ . All components of the null vector are zero  $[\text{latex}]\vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k}[/\text{latex}]$ , so the null vector has no length and no direction.

Two vectors  $[\text{latex}]\vec{A}[/\text{latex}]$  and  $[\text{latex}]\vec{B}[/\text{latex}]$  are equal vectors if and only if their difference is the null vector:

$$[\text{latex}]\vec{0} = \vec{A} - \vec{B} = (A_x\hat{i} - B_x\hat{i}) + (A_y\hat{j} - B_y\hat{j}) + (A_z\hat{k} - B_z\hat{k})$$

$$[\text{latex}]\text{space} = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j} + (A_z - B_z)\hat{k}[/\text{latex}]$$

This vector equation means we must have simultaneously  $[\text{latex}]A_x - B_x = 0[/\text{latex}]$ ,  $[\text{latex}]A_y - B_y = 0[/\text{latex}]$ , and  $[\text{latex}]A_z - B_z = 0[/\text{latex}]$ . Hence, we can write  $[\text{latex}]\vec{A} = \vec{B}[/\text{latex}]$  if and only if the corresponding components of vectors  $[\text{latex}]\vec{A}[/\text{latex}]$  and  $[\text{latex}]\vec{B}[/\text{latex}]$  are equal:

$$[\text{latex}]\vec{A} = \vec{B} \quad \text{if} \quad [\text{latex}]\begin{bmatrix} A_x = B_x \\ A_y = B_y \\ A_z = B_z \end{bmatrix}[/\text{latex}]$$

Two vectors are equal when their corresponding scalar components are equal.

Resolving vectors into their scalar components (i.e., finding their scalar components) and expressing them analytically in vector component form allows us to use vector algebra to find sums or differences of many vectors *analytically* (i.e., using graphical methods). For example, to find the resultant of two vectors  $[\text{latex}]\vec{A}[/\text{latex}]$  and  $[\text{latex}]\vec{B}[/\text{latex}]$ , we simply add them component by component, as follows:

$$\begin{aligned} [\text{latex}]\vec{R} &= \vec{A} + \vec{B} \\ &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) + (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k} \end{aligned}$$

In this way, scalar components of the resultant vector:  $[\text{latex}]\vec{R} = (R_x\hat{i} + R_y\hat{j} + R_z\hat{k})[/\text{latex}]$ .

$$\begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} = \begin{bmatrix} A_x + B_x \\ A_y + B_y \\ A_z + B_z \end{bmatrix}$$

Source: University Physics Volume 1, OpenStax CNX,  
<https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/2-3-algebra-of-vectors/>



### Key Takeaways

*Basically:* Vectors help describe position, forces, and quantities. Vectors use components, magnitude, and direction (unit vector) to do so.

*Application:* A hammock hangs at an angle from the wall. When a person is in the hammock, they are pulling on the wall with a force at an angle. This force vector could be componentized into x and y, using the angle and the weight of the person to calculate it.

*Looking ahead:* The next place vectors will appear is in Moments in 1.6.

# 1.4 Dot Product

A dot product produces a single number to describe the product of two vectors. If you haven't taken linear algebra yet, this may be a new concept. This is a form of multiplication that is used to calculate work, unit vectors, and to find the angle between two vectors.

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$$

A vector can be multiplied by another vector but may not be divided by another vector. There are two kinds of products of vectors used broadly in physics and engineering. One kind of multiplication is a *scalar multiplication of two vectors*. Taking a scalar product of two vectors results in a number (a scalar), as its name indicates. Scalar products are used to define work and energy relations. For example, the work that a force (a vector) performs on an object while causing its displacement (a vector) is defined as a scalar product of the force vector with the displacement vector. A quite different kind of multiplication is a *vector multiplication of vectors*. Taking a vector product of two vectors returns as a result a vector, as its name suggests. Vector products are used to define other derived vector quantities. For example, in describing rotations, a vector quantity called *torque* is defined as a vector product of an applied force (a vector) and its distance from pivot to force (a vector). It is important to distinguish between these two kinds of vector multiplications because the scalar product is a scalar quantity and a vector product is a vector quantity.

Scalar multiplication of two vectors yields a scalar product.

## **Dot Product**

The **scalar product**  $\vec{A} \cdot \vec{B}$  of two vectors  $\vec{A}$  and  $\vec{B}$  is a number defined by the equation:

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \phi$$

where  $\phi$  is the angle between the vectors. The scalar product is also called the **dot product** because of the dot notation that indicates it.

When the vectors are given in their vector component forms:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad \text{and} \quad \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

we can compute their scalar product as follows:

$$\vec{A} \cdot \vec{B}$$

$$\begin{aligned} \vec{B} = (A_x \underline{\hat{i}} + A_y \underline{\hat{j}} + A_z \underline{\hat{k}}) \cdot (B_x \underline{\hat{i}} + B_y \underline{\hat{j}} + B_z \underline{\hat{k}}) \\ = A_x B_x \underline{\hat{i}} \cdot \underline{\hat{i}} + A_x B_y \underline{\hat{i}} \cdot \underline{\hat{j}} + A_x B_z \underline{\hat{i}} \cdot \underline{\hat{k}} \\ + A_y B_x \underline{\hat{j}} \cdot \underline{\hat{i}} + A_y B_y \underline{\hat{j}} \cdot \underline{\hat{j}} + A_y B_z \underline{\hat{j}} \cdot \underline{\hat{k}} \\ + A_z B_x \underline{\hat{k}} \cdot \underline{\hat{i}} + A_z B_y \underline{\hat{k}} \cdot \underline{\hat{j}} + A_z B_z \underline{\hat{k}} \cdot \underline{\hat{k}} \end{aligned}$$

Since scalar products of two different unit vectors of axes give zero, and scalar products of unit vectors with themselves give one, there are only three nonzero terms in this expression. Thus, the scalar product simplifies to:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

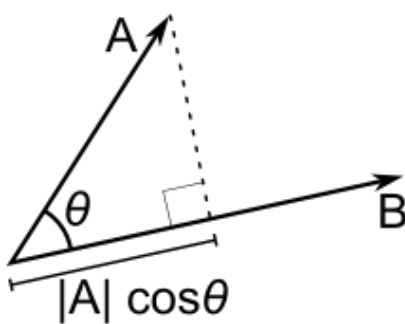
We can use the equation below to find the angle between two vectors. When we divide  $\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \phi$  by  $|\vec{A}| |\vec{B}|$ , we obtain the equation for  $\cos(\phi)$ , into which we substitute the equation from above:

$$\cos \phi = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} = \frac{A_x B_x + A_y B_y + A_z B_z}{|\vec{A}| |\vec{B}|}$$

Angle  $\phi$  between vectors  $\vec{A}$  and  $\vec{B}$  is obtained by taking the inverse cosine of the expression above.

Source: University Physics Volume 1, OpenStax CNX,  
<https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/2-4-products-of-vectors> (many examples at this page).

### But what *IS* it?



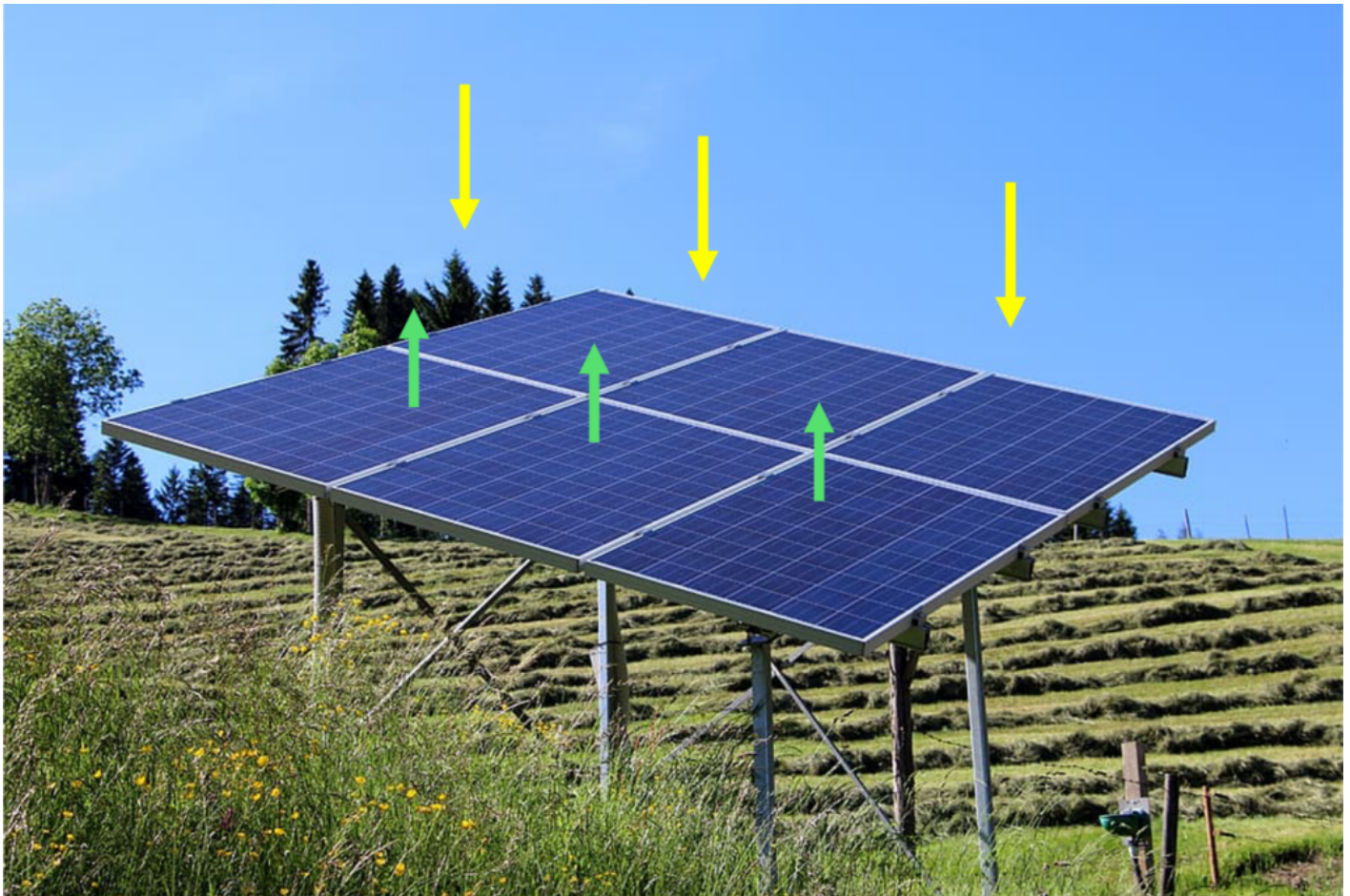
The dot product is the component of vector A along B (  $|A| \cos \theta$  ) times the magnitude (size of B). OR, it's the component of B on A times the magnitude of A. Visually this can be seen in the figure.

There is a nice mathematical proof on page 169 of [Calculus-Based Physics](#).

One neat thing about the dot product is that  $\underline{A} \cdot \underline{B} = \underline{B} \cdot \underline{A}$

An example of a dot product is in a solar panel. To maximize efficiency, the rays coming from the sun

should be perpendicular to the panels, that is, straight on. You could use the dot product between a vector of the sun's rays (yellow in the image below) and the unit vector perpendicular to the surface (green in the image) to calculate what portion of a ray that comes in at an angle produces energy.



Arrows added to photo from Source: <https://www.pxfuel.com/en/free-photo-ouswv>

#### Key Takeaways

*Basically:* Dot product is a method to find a number that is a product of two vectors.

*Application:* Two ropes attached to a sign are being pulled in different directions. To find the angle between them, use the dot product of the two vectors.

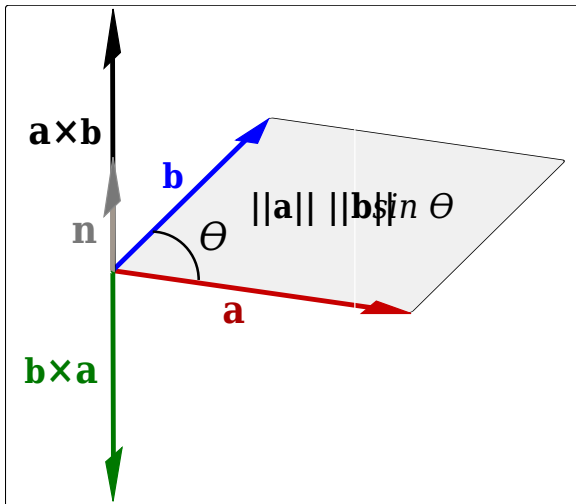
*Looking ahead:* We will use the dot product in Section 2.3 on particle equilibrium equations (and more in dynamics next semester).

Source: [https://en.wikipedia.org/wiki/Dot\\_product#/media/File:Dot\\_Product.svg](https://en.wikipedia.org/wiki/Dot_product#/media/File:Dot_Product.svg)



# 1.5 Cross Products

This is a second way to calculate the product of two vectors. It creates a third vector that is perpendicular to the plane made from the two vectors as shown in the figure .



The black arrow is perpendicular to the grey plane (made from blue and red vectors). This is how you will find the amount of torque created from a force, which we will do many times. Also, unlike the dot product,  $\mathbf{a} \times \mathbf{b}$  is a different direction than  $\mathbf{b} \times \mathbf{a}$ .

$$\begin{aligned} \vec{A} \times \vec{B} = & \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} \\ & \end{aligned}$$

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k}$$

$$|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta$$

The **vector product** of two vectors  $\vec{A}$  and  $\vec{B}$  is denoted by  $\vec{A} \times \vec{B}$  and is often referred to as a **cross product**. The vector product is a vector that has its direction perpendicular to both vectors  $\vec{A}$  and  $\vec{B}$ . In other words, vector  $\vec{A} \times \vec{B}$  is perpendicular to the plane that contains vectors  $\vec{A}$  and  $\vec{B}$ .

Source: University Physics Volume 1, OpenStax CNX,  
<https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/2-4-products-of-vectors/>

Unit vectors allow for a straightforward calculation of the cross product of two vectors under even the most general circumstances, e.g. circumstances in which each of the vectors is pointing in an arbitrary direction in a three-dimensional space. To take advantage of the method, we need to know the cross product of the Cartesian coordinate axis unit vectors  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  with each other. First off, we should note that any vector crossed into itself gives zero. This is evident from the equation:

$$|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta$$

because if  $A$  and  $B$  are in the same direction, then  $\theta = 0^\circ$ , and since  $\sin 0^\circ = 0$ , we have  $|\vec{A} \times \vec{B}| = 0$ . Regarding the unit vectors, this means that:

$$\hat{i} \times \hat{i} = 0 \quad \hat{j} \times \hat{j} = 0 \quad \hat{k} \times \hat{k} = 0$$

Next we note that the magnitude of the cross product of two vectors that are perpendicular to each other is just the ordinary product of the magnitudes of the vectors. This is also evident from the equation:

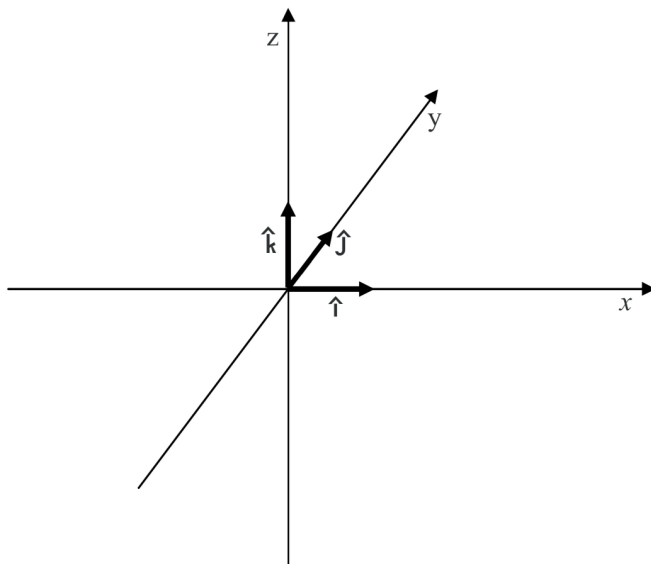
$$|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta$$

because if  $\vec{A}$  is perpendicular to  $\vec{B}$  then  $\theta = 90^\circ$  and  $\sin 90^\circ = 1$  so:

$$|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}|$$

Now if  $A$  and  $B$  are unit vectors, then their magnitudes are both 1, so, the product of their magnitudes is also 1. Furthermore, the unit vectors  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  are all perpendicular to each other so the magnitude of the cross product of any one of them with any other one of them is the product of the two magnitudes, that is, 1.

Now how about the direction? Let's use the right hand rule to get the direction of  $\hat{i} \times \hat{j}$ :



**Figure 1**

[To find the direction, we use the right-hand rule which we will cover more in section 3.1. Here is an overview]. With the fingers of the right hand pointing directly away from the right elbow, and in the same direction as  $\hat{i}$ , (the first vector in “[ $\underline{\hat{i}} \times \underline{\hat{j}}$ ]”) to make it so that if one were to close the fingers, they would point in the same direction as  $\underline{\hat{j}}$ , the palm must be facing in the +y direction. That being the case, the extended thumb must be pointing in the +z direction. Putting the magnitude (the magnitude of each unit vector is 1) and direction (+z) information together we see that:

- $\underline{\hat{i}} \times \underline{\hat{j}} = \underline{\hat{k}}$
- $\underline{\hat{j}} \times \underline{\hat{k}} = \underline{\hat{i}}$
- $\underline{\hat{k}} \times \underline{\hat{i}} = \underline{\hat{j}}$
- $\underline{\hat{j}} \times \underline{\hat{i}} = -\underline{\hat{k}}$
- $\underline{\hat{k}} \times \underline{\hat{j}} = -\underline{\hat{i}}$
- $\underline{\hat{i}} \times \underline{\hat{k}} = -\underline{\hat{j}}$

One way of remembering this is to write

$\underline{\hat{i}}, \underline{\hat{j}}, \underline{\hat{k}}$  twice in succession:

$\underline{\hat{i}}, \underline{\hat{j}}, \underline{\hat{k}}, \underline{\hat{i}}, \underline{\hat{j}}, \underline{\hat{k}}, \underline{\hat{i}}, \underline{\hat{j}}, \underline{\hat{k}}$

Then, crossing any one of the first three vectors into the vector immediately to its right yields the next vector to the right. But crossing any one of the last three vectors into the vector immediately to its left yields the negative of the next vector to the left (left-to-right “+”, but right-to-left “−”).

Now we’re ready to look at the general case. Any vector  $\underline{A}$  can be expressed in terms of unit vectors:

$$\underline{A} = A_x \underline{\hat{i}} + A_y \underline{\hat{j}} + A_z \underline{\hat{k}}$$

Doing the same for a vector  $\underline{B}$  then allows us to write the cross product as:

$$\underline{A} \times \underline{B} = (A_x \underline{\hat{i}} + A_y \underline{\hat{j}} + A_z \underline{\hat{k}}) \times (B_x \underline{\hat{i}} + B_y \underline{\hat{j}} + B_z \underline{\hat{k}})$$

Using the distributive rule for multiplication we can write this as:

$$\underline{A} \times \underline{B} = A_x \underline{\hat{i}} \times (B_x \underline{\hat{i}} + B_y \underline{\hat{j}} + B_z \underline{\hat{k}}) + A_y \underline{\hat{j}} \times (B_x \underline{\hat{i}} + B_y \underline{\hat{j}} + B_z \underline{\hat{k}}) + A_z \underline{\hat{k}} \times (B_x \underline{\hat{i}} + B_y \underline{\hat{j}} + B_z \underline{\hat{k}})$$

$$\underline{A} \times \underline{B} = A_x \underline{\hat{i}} \times B_x \underline{\hat{i}} + A_x \underline{\hat{i}} \times B_y \underline{\hat{j}} + A_x \underline{\hat{i}} \times B_z \underline{\hat{k}} + A_y \underline{\hat{j}} \times B_x \underline{\hat{i}} + A_y \underline{\hat{j}} \times B_y \underline{\hat{j}} + A_y \underline{\hat{j}} \times B_z \underline{\hat{k}} + A_z \underline{\hat{k}} \times B_x \underline{\hat{i}} + A_z \underline{\hat{k}} \times B_y \underline{\hat{j}} + A_z \underline{\hat{k}} \times B_z \underline{\hat{k}}$$

Using, in each term, the commutative rule and the associative rule for multiplication we can write this as:

$$\underline{A} \times \underline{B} = A_x B_x \underline{\hat{i}} \times \underline{\hat{i}} + A_x B_y \underline{\hat{i}} \times \underline{\hat{j}} + A_x B_z \underline{\hat{i}} \times \underline{\hat{k}} + A_y B_x \underline{\hat{j}} \times \underline{\hat{i}} + A_y B_y \underline{\hat{j}} \times \underline{\hat{j}} + A_y B_z \underline{\hat{j}} \times \underline{\hat{k}} + A_z B_x \underline{\hat{k}} \times \underline{\hat{i}} + A_z B_y \underline{\hat{k}} \times \underline{\hat{j}} + A_z B_z \underline{\hat{k}} \times \underline{\hat{k}}$$

$$y_{\underline{j}} \times x_{\underline{k}} + A_{x_{\underline{k}}} y_{\underline{i}} + A_{y_{\underline{k}}} x_{\underline{j}} + A_{z_{\underline{k}}} z_{\underline{k}} \end{aligned} \\$$

Now we evaluate the cross product that appears in each term:

$$\begin{aligned} \vec{A} \times \vec{B} = & A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \\ = & (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k} \end{aligned}$$

Eliminating the zero terms and grouping the terms with  $i$  together, the terms with  $j$  together, and the terms with  $k$  together yields:

$$\begin{aligned} \vec{A} \times \vec{B} = & A_y B_z (\underline{\hat{i}}) + A_z B_y (-\underline{\hat{j}}) + A_x B_z (-\underline{\hat{j}}) \\ & + A_z B_x (\underline{\hat{j}}) + A_x B_y (\underline{\hat{k}}) + A_y B_x (-\underline{\hat{k}}) \end{aligned}$$

Factoring out the unit vectors yields:

$$\begin{aligned} \vec{A} \times \vec{B} = & (\underline{A_y B_z - A_z B_y}) \underline{\hat{i}} + (\underline{A_z B_x - A_x B_z}) \underline{\hat{j}} + (\underline{A_x B_y - A_y B_x}) \underline{\hat{k}} \end{aligned}$$

which can be written on one line as:

$$\vec{A} \times \vec{B} = (\underline{A_y B_z - A_z B_y}) \hat{i} + (\underline{A_z B_x - A_x B_z}) \hat{j} + (\underline{A_x B_y - A_y B_x}) \hat{k}$$

This is our end result. We can arrive at this result much more quickly if we borrow a tool from that branch of mathematics known as linear algebra (the mathematics of matrices).

We form the  $3 \times 3$  matrix:

$$\begin{bmatrix} \underline{\hat{i}} & \underline{\hat{j}} & \underline{\hat{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

by writing  $i, j, k$  as the first row, then the components of the first vector that appears in the cross product as the second row, and finally the components of the second vector that appears in the cross product as the last row. It turns out that the cross product is equal to the determinant of that matrix. We use absolute value signs on the entire matrix to signify “the determinant of the matrix.” So we have:

$$\underline{\hat{i}} \otimes \underline{\hat{j}} \otimes \underline{\hat{k}} \\ A_x \otimes A_y \otimes A_z \\ B_x \otimes B_y \otimes B_z$$

To take the determinant of a  $3 \times 3$  matrix you work your way across the top row. For each element in that row you take the product of the elements along the diagonal that extends down and to the right, minus the product of the

elements down and to the left; and you add the three results (one result for each element in the top row) together. If there are no elements down and to the appropriate side, you move over to the other side of the matrix (see below) to complete the diagonal.

For the first element of the first row, the  $\hat{i}$ , take the product down and to the right,

$$\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{array}$$

( this yields  $iA_yB_z$ ) minus the product down and to the left

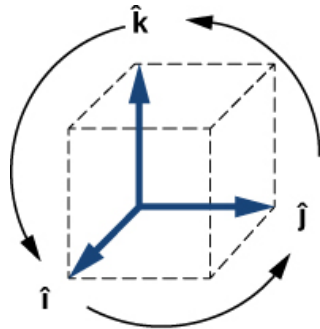
$$\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{array}$$

( the product down-and-to-the-left is  $iA_zB_y$ ).

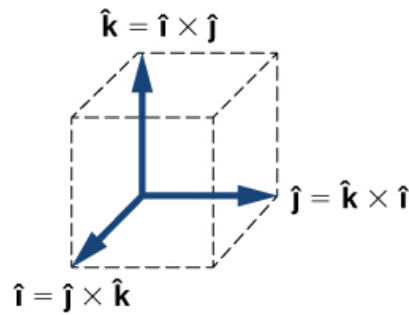
For the first element in the first row, we thus have:  $\underline{\hat{i}}[A_yB_z - A_zB_y]$  which can be written as:  $(A_yB_z - A_zB_y)\underline{\hat{i}}$ . Repeating the process for the second and third elements in the first row (the  $\hat{j}$  and the  $\hat{k}$ ) we get  $(A_zB_x - A_xB_z)\underline{\hat{j}}$  and  $(A_xB_y - A_yB_x)\underline{\hat{k}}$  respectively. Adding the three results, to form the determinant of the matrix results in:

$$\underline{\vec{A}} \times \underline{\vec{B}} = (A_yB_z - A_zB_y)\underline{\hat{i}} + (A_zB_x - A_xB_z)\underline{\hat{j}} + (A_xB_y - A_yB_x)\underline{\hat{k}}$$

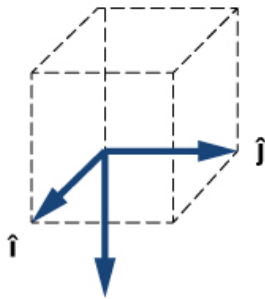
as we found before, “the hard way.” Below the diagram shows the direction of each part of the cross product:



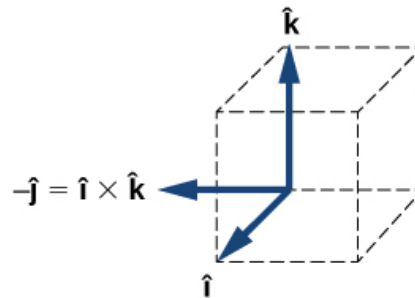
(a)



(b)



(c)



(d)

Source: Calculus Based Physics, Jeffrey W. Schnick,

<https://openlibrary.ecampusontario.ca/catalogue/item/?id=ce74a181-ccde-491c-848d-05489ed182e7> page 136-141

### Key Takeaways

*Basically:* Cross product is a method to find a vector that is a product of two vectors, perpendicular to the plane created from the two vectors.

*Application:* While bicycling, the force of my foot pushing on the pedal produces the most rotation when the force is 90 degrees (perpendicular) from the surface of the pedal. The pedals themselves rotate so I can change the angle I am pushing with, otherwise when the pedal is at the bottom, it would be very difficult to produce forward motion.

*Looking ahead:* In the next section, 1.6 on Torque, we will use the cross product to find the moment produced from forces.

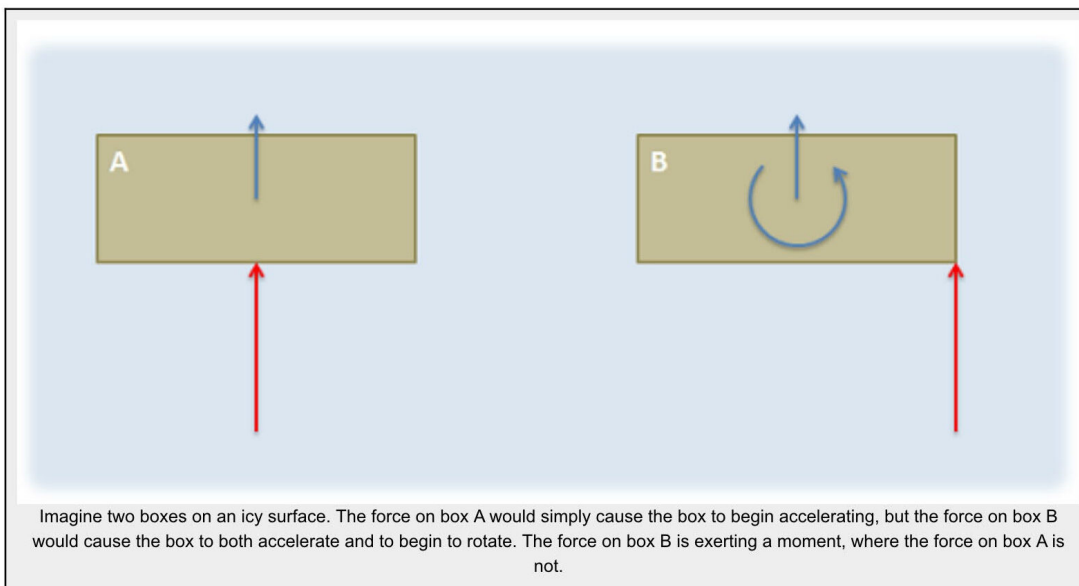
<https://commons.wikimedia.org/wiki/File:Cross-product-with-area.svg>

# 1.6 Torque/Moment

## 1.6.1 Moments

Moments are created by a force acting a distance from the center of rotation. There are three ways to calculate moments: scalar, vector, and using the right hand rule. The first two methods will be presented in this section and the third way will be discussed in section 3.1.3 on the right-hand rule.

A **moment** (sometimes called a torque) is defined as the “tendency of a force to rotate a body”. Where forces cause linear accelerations, moments cause **angular accelerations**. In this way moments, can be thought of as twisting forces.



### The Vector Representation of a Moment:

Moments, like forces, can be represented as vectors and have a magnitude, a direction, and a “point of application”. For moments however a better name for the point of application is the **axis of rotation**. This will be the point or axis about which we will determine all the moments.

### Magnitude:

The magnitude of a moment is the degree to which the moment will cause angular acceleration in the body it is acting on. It is represented by a scalar (a single number). The magnitude of the moment can be thought of as the strength of the twisting force exerted on the body. When a moment is represented as a vector, the magnitude of the moment is usually explicitly labeled, though the length of the moment vector also often corresponds to the relative magnitude of the moment.

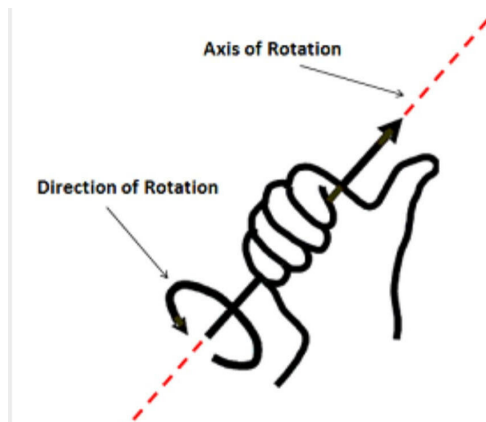
The magnitude of the moment is measured in units of force times distance. The standard metric units for the magnitude of moments are newton meters, and the standard English units for a moment are foot pounds.

$$M = F \cdot d \quad \text{Metric: N} \cdot \text{m} \quad \text{English: lb} \cdot \text{ft}$$

### Direction:

In a two dimensional problem, the direction can be thought of as a scalar quantity corresponding to the direction of rotation the moment would cause. A moment that would cause a counter-clockwise rotation is a positive moment and a moment that would cause a clockwise rotation is a negative moment.

In a three dimensional problem however, a body can rotate about an axis in any direction. If this is the case we need a vector to represent the direction of the moment. The direction of the moment vector will line up with the axis of rotation that moment would cause, but to determine which of the two directions we can use along that axis we have available we use the right hand rule. To use the right hand rule, align your right hand as shown so that your thumb lines up with the axis of rotation for the moment and your curled fingers point in the direction of rotation for your moment. If you do this, your thumb will be pointing in the direction of the moment vector.

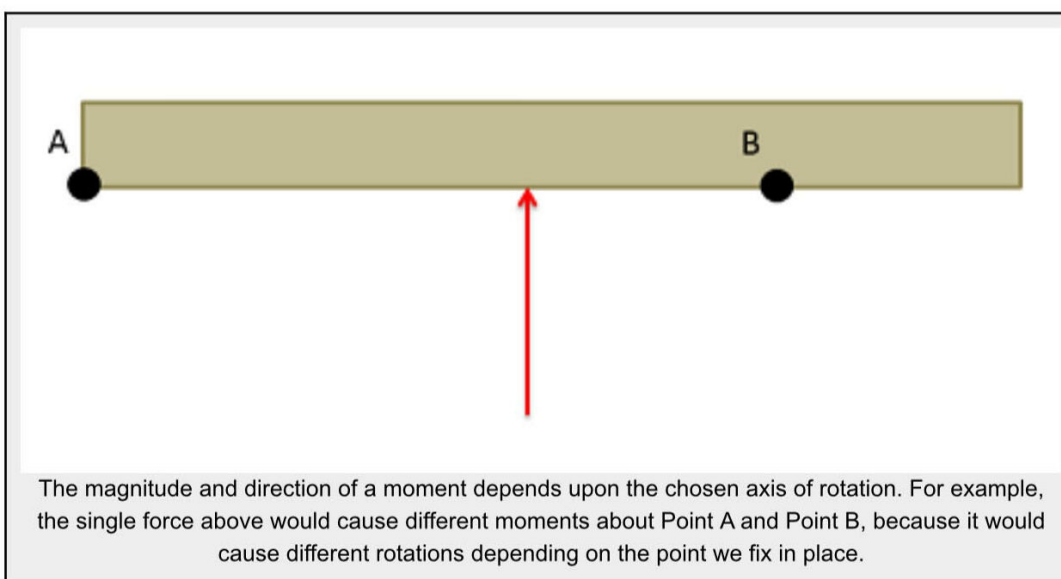


If we look back to two dimensional problems, all rotations occur about an axis pointing directly into or out of the page (the z axis). Using the right hand rule, counter-clockwise rotations are represented by a vector in the positive z direction and clockwise rotations are represented by a vector in the negative z direction.

### Axis of Rotation:

In engineering statics problems we can choose any point/axis as the axis of rotation. The choice of this point will affect the magnitude and direction of the resulting moment however, and the moment is only valid about that point.





Though we can take the moment about any point in a statics problem, if we are **adding** together the moments from multiple forces, all the moments must be taken about a **common axis of rotation**. Moments taken about different points cannot be added together to find a 'net moment'

Additionally, if we move into the subject of dynamics, where bodies are moving, we will want to relate moments to angular accelerations. For this to work, we will need to take the moments either about a single point that does not move (such as the hinge on a door) or we will need to take the moments about the center of mass of the body. Summing moments about other axes of rotation will not result in valid calculations.

### Calculating Moments:

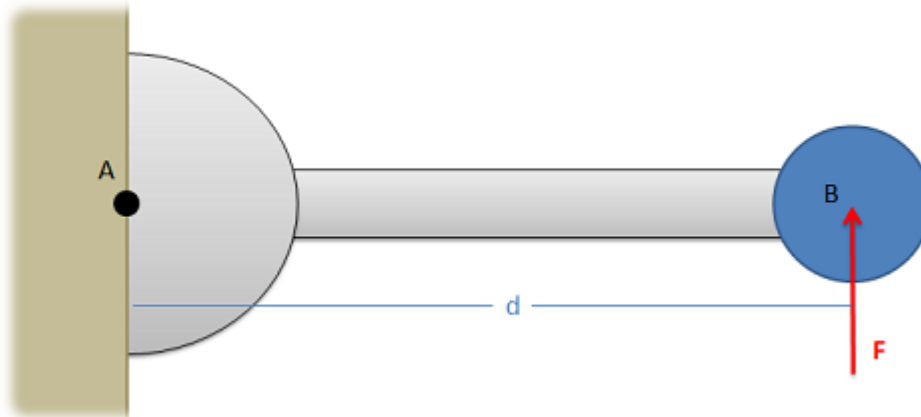
To calculate the moment that a force exerts on a body, we will have two main options: **scalar methods** and **vector methods**. Scalar methods are generally faster for two dimensional problems where a body can only rotate clockwise or counter-clockwise, while vector methods are generally faster for three dimensional problems where the axis of rotation is more complex.

Given any point on an extended body, if there is a force acting on that body that does not travel through that point, then that force will cause a moment about that point. As discussed on the moments page, a moment is a force's tendency to cause rotation.

Source: Engineering Mechanics, Jacob Moore et al.,  
[http://mechanicsmap.psu.edu/websites/1\\_mechanics\\_basics/1-5\\_moments/moments.html](http://mechanicsmap.psu.edu/websites/1_mechanics_basics/1-5_moments/moments.html)

## 1.6.2 Scalar Method in 2 Dimensions

In discussing how to calculate the moment of a force about a point via scalar quantities, we will begin with the example of a force on a simple lever as shown below. In this simple lever there is a force on the end of the lever, distance  $d$  away from the center of rotation for the lever (point A) where the force has a magnitude  $F$ .

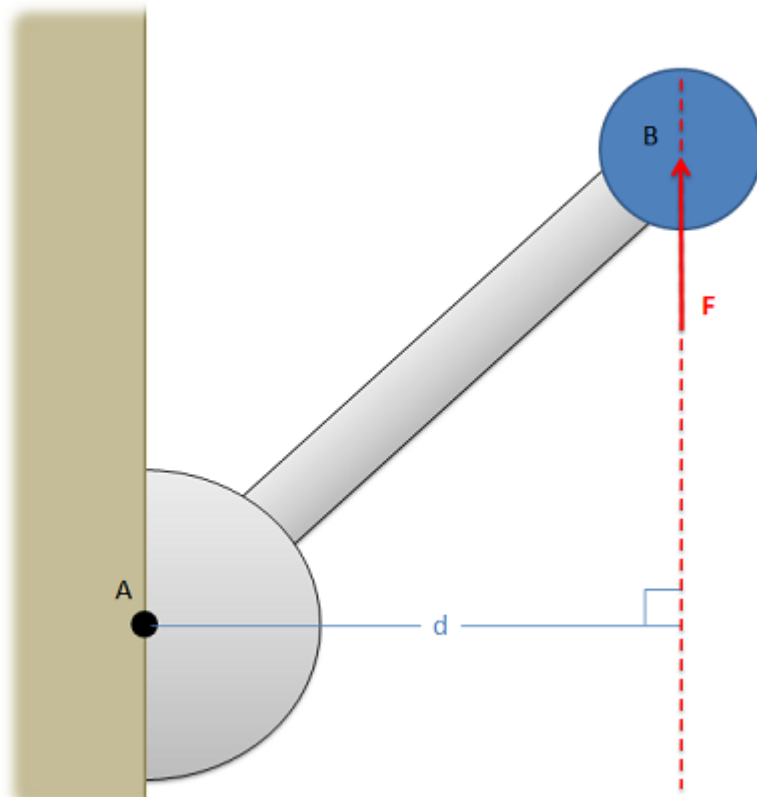


When using scalar quantities, the magnitude of the moment will be equal to the perpendicular distance between the line of action of the force and the point we are taking the moment about.

$$M = Fd$$

To determine the sign of the moment, we determine what type of rotation the force would cause. In this case we can see that the force would cause the lever to rotate counterclockwise about point A. Counterclockwise rotations are caused by positive moments while clockwise rotations are caused by negative moments.

Another important factor to remember is that the value  $d$  is the perpendicular distance from the force to the point we are taking the moment about. We could measure the distance from point A to the head of the force vector, or the tail of the force vector, or really any point along the line of action of force  $F$ . The distance we need to use for the scalar moment calculation however is the shortest distance between the point and the line of action of the force. This will always be a line perpendicular to the line of action of the force, going to the point we are taking the moment about.



Source: Engineering Mechanics, Jacob Moore et al.,  
[http://mechanicsmap.psu.edu/websites/3\\_equilibrium\\_rigid\\_body/3-1\\_moment\\_scalar/moment\\_scalar.html](http://mechanicsmap.psu.edu/websites/3_equilibrium_rigid_body/3-1_moment_scalar/moment_scalar.html)

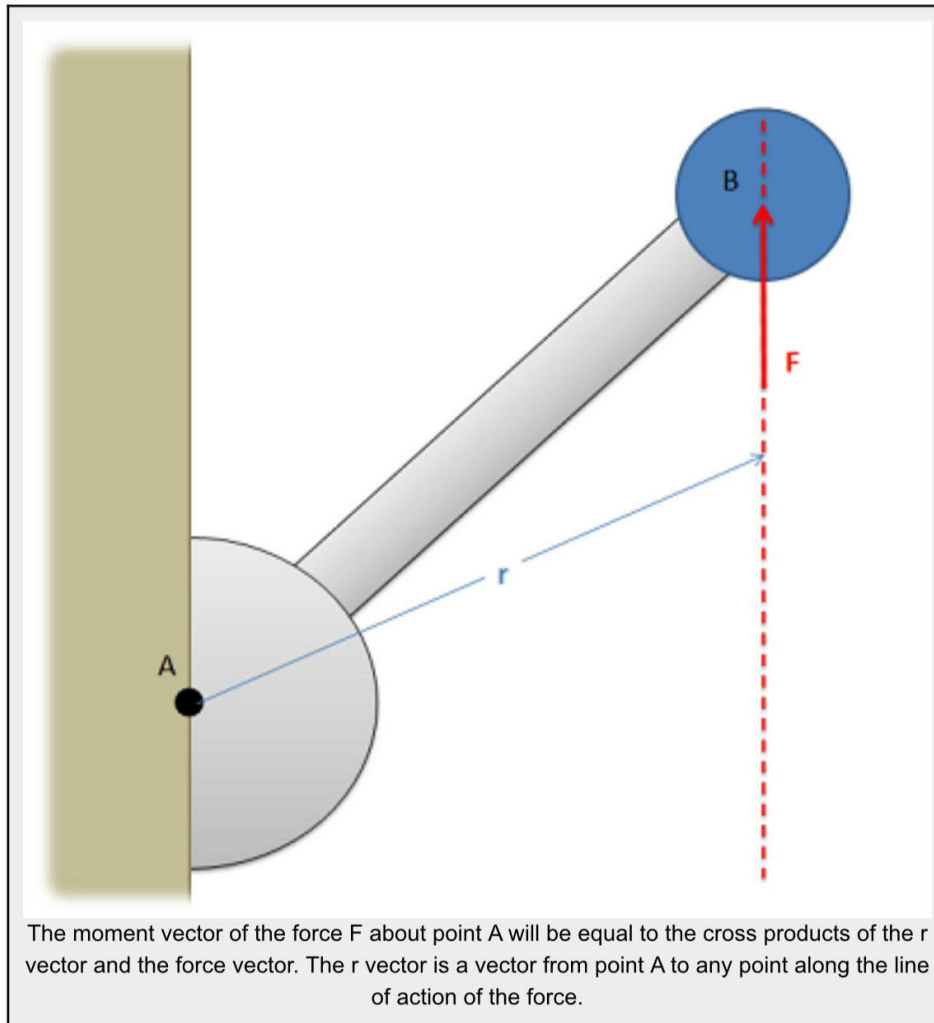
If the position vector and force aren't at exactly 90 degrees, the equation  $T = |\mathbf{r}| |\mathbf{F}| \sin \Theta$  can be helpful and still produce a scalar number for moment or torque. This is good if we had really simple systems, such as opening a door or spinning a top. What happens when you have a rotating door and a few people are trying to go through it? Or a force that isn't exactly along two axes? To solve that, we'll use vectors as shown in the next section.

## 1.6.3 Vector Method in 3 Dimensions

An alternative to calculating the moment via scalar quantities is to use the **vector method** or **cross product method**. For simple two dimensional problems, using scalar quantities is usually easier, but for more complex problems, using the cross product method is usually easier. The cross product method for calculating moments says that the moment vector of a force about a point will be equal to the cross product of a vector  $\mathbf{r}$  from the point to anywhere on the line of action of the force and the force vector itself.

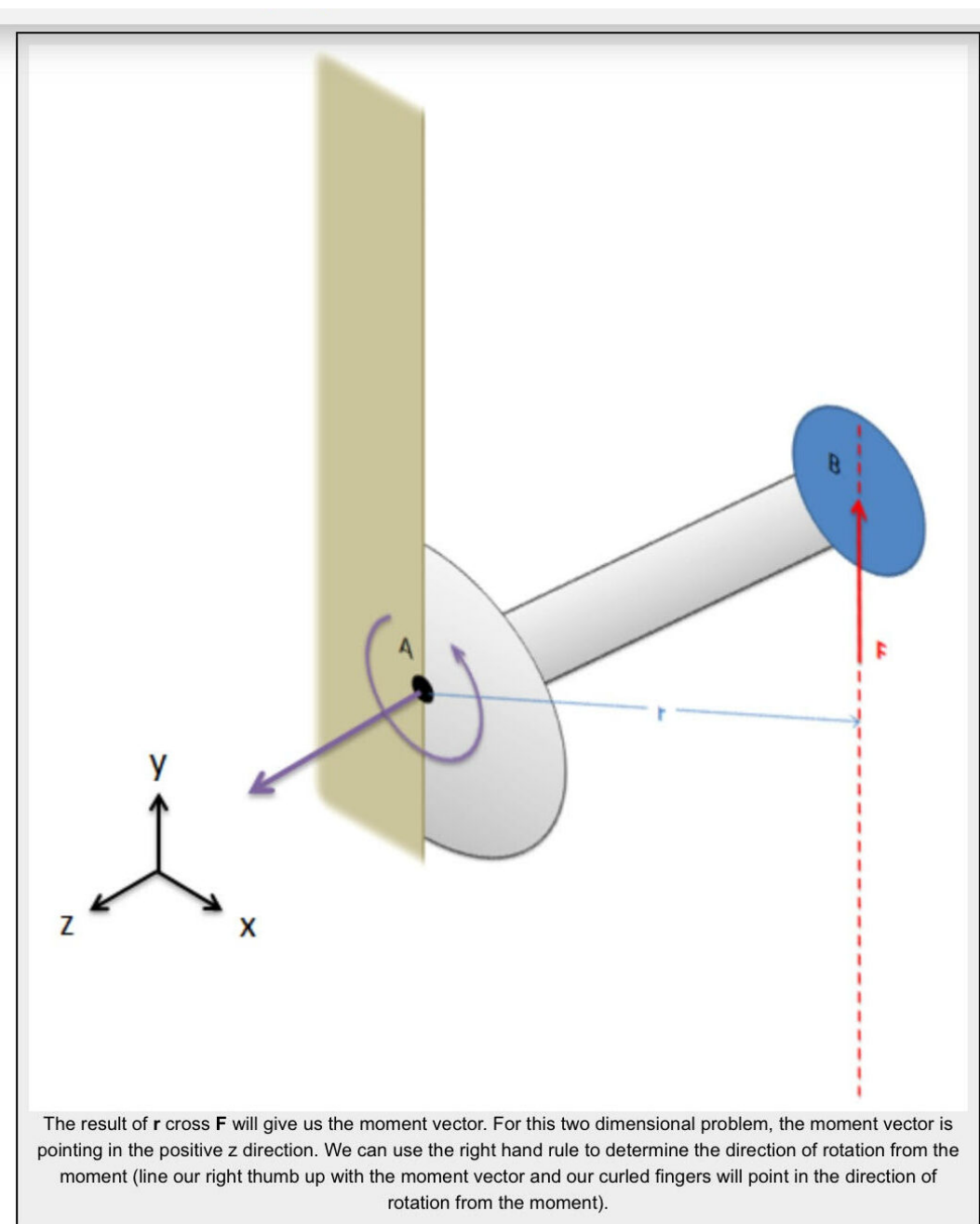
$$\vec{M} = \vec{r} \times \vec{F}$$

A big advantage of this method is that  $\vec{r}$  does not have to be the shortest distance between the point and the line of action, it goes from the point to any part of the line of action. For any problem, there are many possible  $\vec{r}$  vectors, though because of the way the cross product works, they should all result in the same moment vector in the end.



It is important to note here that all quantities ( $\vec{r}$ ,  $\vec{F}$  and  $\vec{M}$ ) are vectors. Before you can solve for the cross product, you will need to write out  $\vec{r}$  and  $\vec{F}$  in vector component form. Also, even for two dimensional problems, you will need to write out all three components of the  $\vec{r}$  and  $\vec{F}$  vectors. For two dimensional problems the  $z$  components of the  $\vec{r}$  and  $\vec{F}$  vectors will simply be zero, but those values are necessary for the calculations.

The moment vector you get will line up with the axis of rotation for the moment, where you can use the right hand rule to determine if the moment is going clockwise or counterclockwise about that axis.



Finally, it is also important to note that cross product, unlike multiplication, is not commutative. This means that the order of the vectors matters, and  $\mathbf{r} \times \mathbf{F}$  will not be the same as  $\mathbf{F} \times \mathbf{r}$ . It is important to always use  $\mathbf{r} \times \mathbf{F}$  when calculating moments.

Source: Engineering Mechanics, Jacob Moore et al.,  
[http://mechanicsmap.psu.edu/websites/3\\_equilibrium\\_rigid\\_body/moment\\_vector/momentvector.html](http://mechanicsmap.psu.edu/websites/3_equilibrium_rigid_body/moment_vector/momentvector.html)

Whether you use scalars:  $M = |\mathbf{r}| |\mathbf{F}| \sin\theta$  or vectors:  $\mathbf{M} = \mathbf{r} \times \mathbf{F}$  you can solve most moment/torque problem. The scalar method is faster for 2-d problems, especially if the vectors are at 90 degree angles

from each other ( $\sin 90^\circ = 1$ ). The vector method is more robust, especially if there are additional angles involved. There is the potential to make errors, so it's recommended to use the Step 6 Review step to try multiple methods to ensure your answer is correct.

See the examples in section 1.8 as many of them concern moments.

#### Key Takeaways

*Basically:* There are three methods to calculate moments, two of which were discussed here. Moments or Torque is created by a force acting some distance from an axis of rotation.

*Application:* When you are opening a heavy door, you push on the door. If you push closer to the axis of rotation, you'll need a bigger force to make it move. If you push further away from the axis (so  $r$  is bigger), the force can be smaller to make the same motion occur.

*Looking ahead:* Moments apply when rigid bodies are involved, so we'll pick up moments again in Chapter 3.1.3, when you'll learn the third way to calculate moments (torque).

# 1.7 Problem Solving Process

Learning how to use a structured problem solving process will help you to be more organized and support your future courses. Also, it will train your brain how to approach problems. Just like basketball players practice jump shots over and over to train their body how to act in high pressure scenarios, if you are comfortable and familiar with a structured problem solving process, when you're in a high pressure situation like a test, you can just jump into the problem like muscle memory.

## 6 Step Problem Solving Method:

### 1. Problem

- Write out the answer with all necessary information that is given to you. It feels like it takes forever, but it's important to have the problem and solution next to each other.

### 2. Draw

- Draw the problem, this is usually a free-body diagram (don't forget a coordinate frame). Eventually, as you get further into the course, you might need a few drawings. One would be a quick sketch of the problem in the real world, then modelling it into a simplified engineering drawing, and finally the free-body diagram.

### 3. Known and Unknowns

- Write out a list of the known/given values with the *variable* and *unit*, i.e.  $m = 14 \text{ kg}$  (variable = number unit)
- Write out a list of the unknown values that you will have to solve for in order to solve the problem
- You can also add any assumptions you made here that change the problem.
- Also state any constants, i.e.  $g = 32.2 \text{ ft/m}^2$  or  $g = 9.81 \text{ m/s}^2$
- This step helps you to have all of the information in one place when you solve the problem. It's also important because each number should include units, so you can see if the units match or if you need to convert some numbers so they are all in English or SI. This also gives you the variables side by side to ensure they are unique (so you don't accidentally have 2 'd' variables and can rename one with a subscript).

#### 4. Approach

- Write a simple sentence or phrase explaining what method/approach you will be using to solve the problem.
- For example: 'use method of joints', or equilibrium equations for a rigid body, MMOI for a certain shape, etc.
- This is going to be more important when you get to the later chapters and especially next semester in Dynamics where you can solve the same problem many ways. Might as well practice now!

#### 5. Analysis

- This is the actual solving step. This is where you show all the work you have done to solve the problem.
- When you get an answer, restate the variable you are solving for, include the unit, and put a box around the answer.

#### 6. Review

- Write a simple sentence explaining why (or why not) your answer makes sense. Use logic and common sense for this step.
- When possible, use a second quick numerical analysis to verify your answer. This is the "gut check" to do a quick calculation to ensure your answer is reasonable.
- This is the most confusing step as students often don't know what to put here and up just writing 'The number looks reasonable'. This step is vitally important to help you learn how to think about your answer. What does that number mean? What is it close to? For example, if you find that  $x = 4000$  m, that's a very large distance! In the review, I would say, 'the object is 4 km long which is reasonable for a long bridge'. See how this is compared to something similar? Or you could do a second calculation to verify the number is correct, such as adding up multiple parts of the problem to confirm the total length is accurate i.e. ' $x + y + z = \text{total}$ , yes it works!'

Additional notes for this course:

- It's important to include the **number** and **label** the steps so it's clear what you're doing, as shown in the example below.
- It's okay if you make mistakes, just put a line through it and keep going.
- Remember your header should include your name, the page number, total number of pages, the course number, and the assignment number. If a problem spans a number of pages, you should include it in the header too.



Libby Osgood

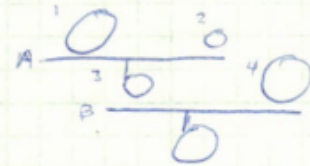
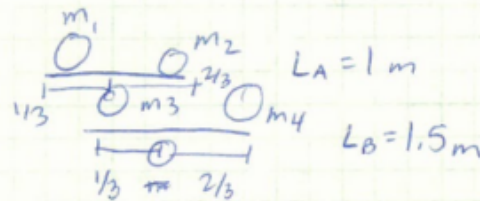
Sept 5, 2019

ENGN 1230: Class Example

1/

1. Problem

A set of rocks are balanced in two levels, what is the mass of the rock on the right to make the system balance?

2. Draw3. Knowns

$$\begin{aligned} m_1 &= 2 \text{ kg} \\ m_2 &= 1 \text{ kg} \\ m_3 &= 1 \text{ kg} \\ g &= 9.8 \text{ m/s}^2 \end{aligned}$$

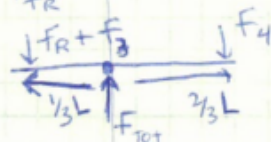
Assume A + B are massless

Un known $m_4$ 4. Approach torque, compare masses, equilibrium equations5. Analysis

For platform A

$$F_R = F_1 + F_2 = (m_1 + m_2)g$$

For platform B



$$\text{Torque} = F \cdot \text{distance}$$

$$\frac{1}{3} L (F_R + F_3) = \frac{2}{3} L F_4$$

$$\frac{1}{3} L (m_1 + m_2 + m_3)g = \frac{2}{3} L m_4 g$$

$$m_1 + m_2 + m_3 = 2 m_4$$

$$m_4 = \frac{(2 \text{ kg} + 1 \text{ kg} + 1 \text{ kg})}{2} = 2 \text{ kg}$$

$$\boxed{m_4 = 2 \text{ kg}}$$

6. Review

There is twice the distance, so it makes sense that the mass is half as much. Also, it's the same mass as the other big rock.

9-0235 — 50 SHEETS — 5 SQUARES  
9-0236 — 100 SHEETS — 5 SQUARES  
9-0237 — 200 SHEETS — 5 SQUARES  
9-0137 — 200 SHEETS — FILLER

COMET

### Key Takeaways

*Basically:* Use a 6-step structured problem solving process: 1. Problem, 2. Draw, 3. Known & Unknown, 4. Approach, 5. Analysis (Solve), 6. Review

*Application:* In your future job there is likely a structure for analysis reports that will be used. Each company has a different approach, but most have a standard that should be followed. This is good practice.

*Looking ahead:* This will be part of every homework assignment.

Written by Gayla & Libby

# 1.8 Examples

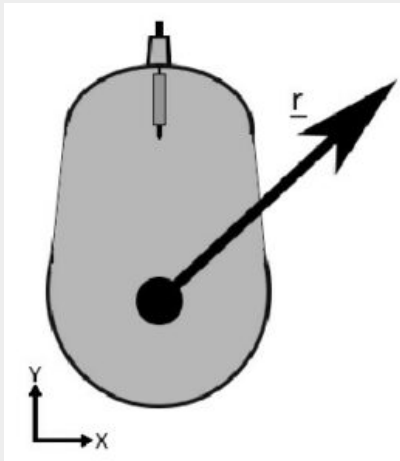
Here are examples from Chapter 1 to help you understand these concepts better. These were taken from the real world and supplied by FSDE students in Summer 2021. If you'd like to submit your own examples, please send them to:

[eosgood@upei.ca](mailto:eosgood@upei.ca)

## Example 1.8.1: Vectors, Submitted by Tyson Ashton-Losee

### 1. Problem

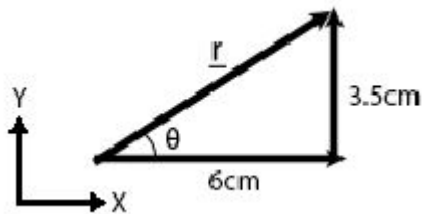
After a long day of studying, a student sitting at their computer moves the cursor from the bottom left of the screen to the top right in order to close a web browser. The computer mouse was displaced 6 cm along the x-axis and 3.5 cm along the y-axis. Draw the resultant vector and calculate the distance traveled.





Source:  
<https://www.flickr.com/photos/dejankrsmannovic/33218207918>

## 2. Draw



## 3. Knowns and Unknowns

Known:

$$x = 6 \text{ cm}$$

$$y = 3.5 \text{ cm}$$

Unknown:  $r, \theta$

## 4. Approach

Use SOH CAH TOA, first find  $\theta$ , then  $r$

## 5. Analysis

```
\begin{aligned}
&\tan \theta = \frac{y}{x} \\
&\tan \theta = \frac{3.5 \text{ cm}}{6 \text{ cm}} \\
&\theta = \tan^{-1} \left( \frac{3.5}{6} \right) \\
&\theta = 30.256^\circ \\
&\sin \theta = \frac{y}{r} \\
&r = \frac{y}{\sin \theta} \\
&r = \frac{3.5 \text{ cm}}{\sin (30.256^\circ)} \\
&r = 6.946 \text{ cm}
\end{aligned}
```

```
&r=6.9 \mathrm{\sim cm}
\end{aligned}
```

## 6. Review

It makes sense that the angle is less than 45, because  $y$  is smaller than  $x$ . Also, if you use Pythagorean theorem to find  $r$ , you get the same answer.

# Example 1.8.2: Vectors, Submitted by Brian MacDonald

## 1. Problem

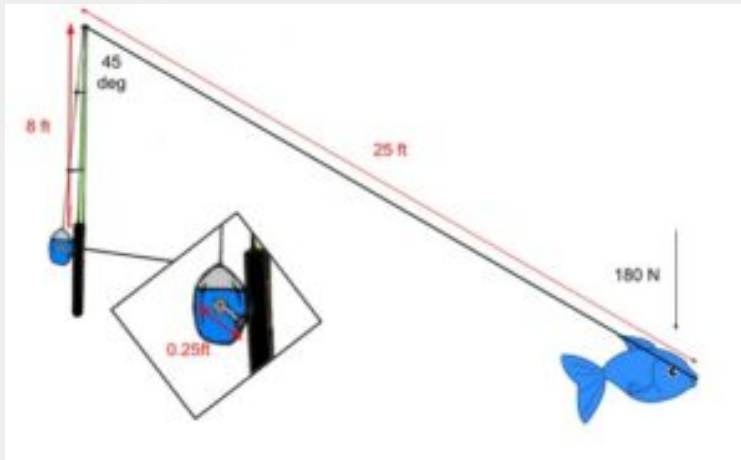
Mark is fishing in the ocean with his favourite fishing rod. The distance between the tip of the rod and the reel is 8 ft and the length of the reel handle is 0.25 ft. The angle between the fishing rod and fishing line is 45 degrees. If Mark catches a fish when 25 ft of the fishing line is released while the fish is diving down with a force of 180 N, how much force does Mark need to apply (push down) to the reel handle to bring in the fish? Draw the position vector of the fish relative to the reel.

Assumptions:

- Mark can reel in the fish when he generates more torque with the handle than the amount of torque that the fish is applying to the reel while pulling on the line.
- The fishing line comes out of the reel in a straight line at a 90-degree angle.

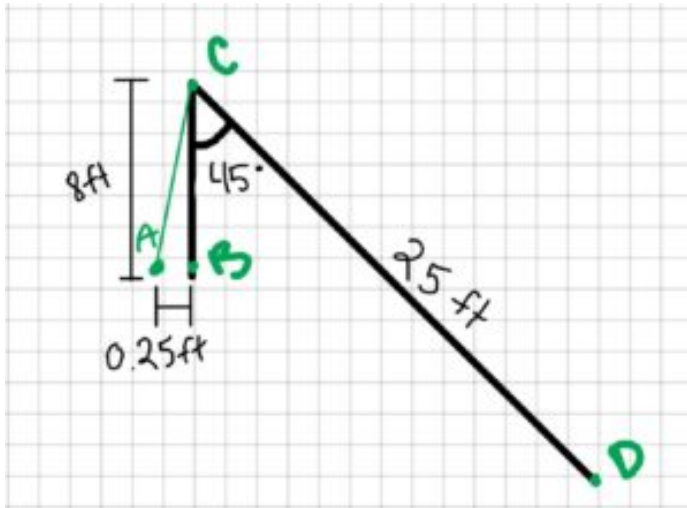


Source: <https://commons.wikimedia.org/wiki/File:Deepsea.JPG>

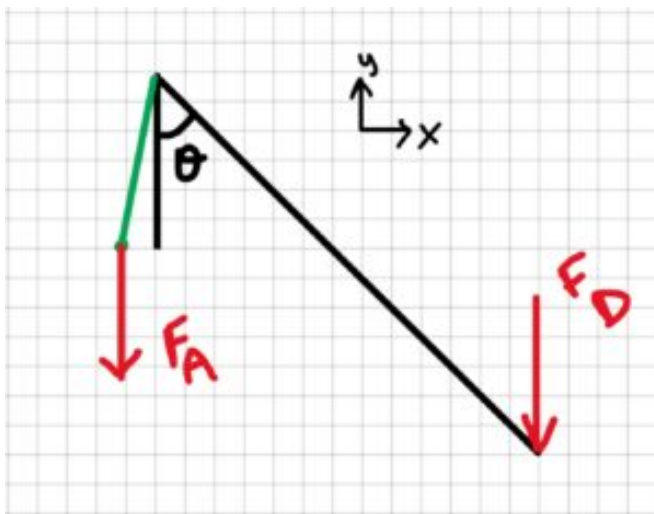


## 2. Draw

Sketch:



Free-body diagram:



### 3. Knowns and Unknowns

Known:

$$\begin{aligned} r_{AB} &= 0.25 \text{ ft} \\ r_{BC} &= 8 \text{ ft} \\ r_{CD} &= 25 \text{ ft} \\ F_D &= 180 \text{ N} \\ \theta &= 45^\circ \end{aligned}$$

Unknown:  $F_A$ , vector  $\underline{r}_{AD}$

### 4. Approach

Convert inches to meters, then use the below equation.

$$T = |r| * |F| * \sin \theta$$

### 5. Analysis

Step 1: convert inches to meters

$$\begin{aligned} &25 \text{ m} * \frac{12 \text{ in}}{1 \text{ m}} * \frac{2.54}{100} \end{aligned}$$

$$\begin{aligned} & \left( \frac{1 \text{ m}}{100 \text{ cm}} \right) = 7.62 \text{ m} \\ & 0.25 \text{ ft} \cdot \left( \frac{12 \text{ in}}{1 \text{ ft}} \right) \cdot \left( \frac{2.54 \text{ cm}}{1 \text{ in}} \right) = 0.0762 \text{ m} \\ & \end{aligned}$$

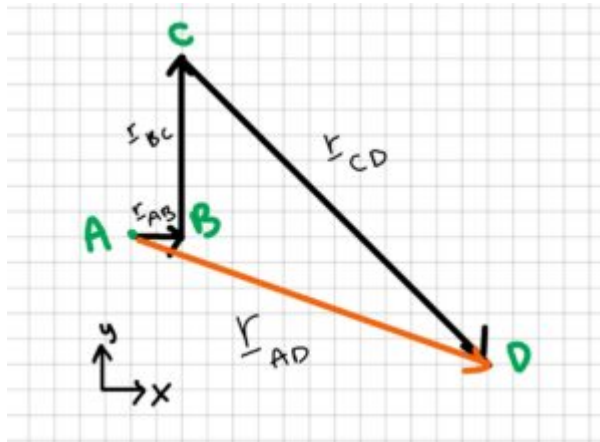
Step 2: solve for  $T_D$

$$T_D = |r_{CD}| \cdot |F_D| \cdot \sin \theta \quad T_D = (7.62 \text{ m})(180 \text{ N}) \sin(45^\circ) \quad T_D = 969.86766 \text{ Nm}$$

Step 3: Solve for  $F_A$

$$\begin{aligned} T_A &= |r_{AB}| \cdot |F_A| \cdot \sin \theta \quad \text{Assume } T_A = T_D \\ F_A &= \frac{T_D}{|r_{AB}| \cdot \sin \theta} \quad F_A = \frac{969.86766 \text{ Nm}}{0.0762 \text{ m} \cdot \sin(45^\circ)} \\ F_A &= 17,999.998 \text{ N} \quad F_A = 18,000 \text{ N} \end{aligned}$$

Vector  $r_{AD}$ :



$$\begin{aligned} \vec{r}_{AD} &= \vec{r}_{AB} + \vec{r}_{BC} + \vec{r}_{CD} \\ &= \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} \text{ ft} + \begin{bmatrix} 0 \\ 8 \end{bmatrix} \text{ ft} + \begin{bmatrix} 25 \sin 45^\circ \\ -25 \cos 45^\circ \end{bmatrix} \text{ ft} \\ &= \begin{bmatrix} 17.93 \\ -9.68 \end{bmatrix} \text{ ft} \end{aligned}$$

## 6. Review

The answer, though yielding a very large number, appears to be correct from the information given. 18,000 N of force is the amount of force Mark would need to apply the reel handle to generate the same amount of force that the fish creates. 18,000 N in reality is too much for one to generate but also in real scenario's one would not have to generate the same amount of force to reel in the fish to reel gearing, the amount of torque generated by the fishing rod itself and etc. In other words 18 000 N of force is too high in a real scenario but with the assumptions given in the problem, the number seems reasonable. The answer also has the correct unit, N.



## Example 1.8.3: Dot product and cross product, submitted by Anonymous ENGN 1230 Student

### 1. Problem

$$\underline{a} = [6\ \ 5\ \ 3] \quad \underline{b} = [8\ \ 1\ \ 3]$$

- a) Find  $6b$
- b) Find  $a \cdot b$
- c) Find  $a \times b$
- d) Find  $2a \times b$

### 2. Draw

n/a

### 3. Knowns and Unknowns

Known:  $\underline{a}$ ,  $\underline{b}$

Unknowns: a)  $6b$ , b)  $a \cdot b$ , c)  $a \times b$ , d)  $2a \times b$

### 4. Approach

Use dot product, cross product equations

### 5. Analysis

Part a:

$$6\underline{b} = 6[8\ \ 1\ \ 3] = [48\ \ 6\ \ 18]$$

Part b:

$$\underline{a} \cdot \underline{b} = [6\ \ 5\ \ 3] \cdot [8\ \ 1\ \ 3] = 6 \cdot 8 + 5 \cdot 1 + 3 \cdot 3 = 48 + 5 + 9 = 62$$

Part c:

```


$$\underline{\hat{a}} \times \underline{\hat{b}} = \begin{bmatrix} \underline{\hat{i}} & \underline{\hat{j}} & \underline{\hat{k}} \\ 6 & 5 & 3 \\ 8 & 1 & 3 \end{bmatrix}$$


$$= \begin{bmatrix} (5 \cdot 3 - 3 \cdot 1) \underline{\hat{i}} - (6 \cdot 3 - 3 \cdot 8) \underline{\hat{j}} + (6 \cdot 1 - 5 \cdot 8) \underline{\hat{k}} \\ 12 \underline{\hat{i}} + 6 \underline{\hat{j}} - 34 \underline{\hat{k}} \end{bmatrix}$$


```

Part d:

```


$$\underline{\hat{a}} = 2 \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 6 \end{bmatrix}, \underline{\hat{b}} = \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}$$


$$\underline{\hat{a}} \times \underline{\hat{b}} = \begin{bmatrix} \underline{\hat{i}} & \underline{\hat{j}} & \underline{\hat{k}} \\ 12 & 10 & 6 \\ 8 & 1 & 3 \end{bmatrix}$$


$$= \begin{bmatrix} (10 \cdot 3 - 6 \cdot 1) \underline{\hat{i}} - (12 \cdot 3 - 6 \cdot 8) \underline{\hat{j}} + (12 \cdot 1 - 10 \cdot 8) \underline{\hat{k}} \\ 24 \underline{\hat{i}} + 12 \underline{\hat{j}} - 68 \underline{\hat{k}} \end{bmatrix}$$


```

## 6. Review

The answer to part d is double the answer for part c, which makes sense. It also makes sense that the answers to b, c, and d have values in three directions, while a only has magnitude.

# Example 1.8.4: Torque, Submitted by Luke McCarvill

## 1. Problem

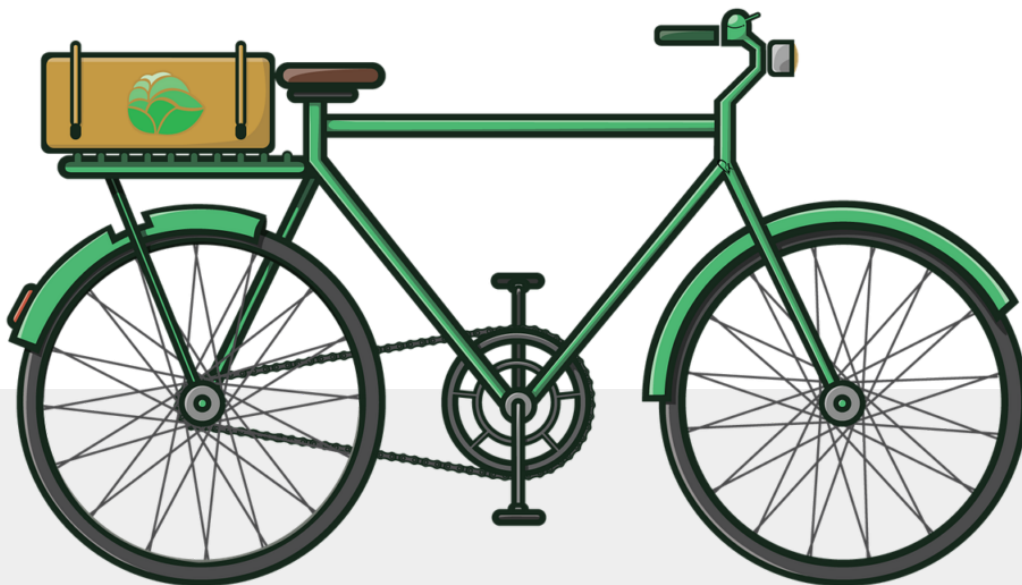
To start riding her bicycle, Jane must push down on one of her bike's pedals which are on 16 centimeter long crank arms. Jane can push directly downwards with her legs with a force of 100N. Jane notices that the pedal's starting position can sometimes make it more or less useful in generating torque.

a) What is the ideal angle that Jane's bike pedal should be at in order to generate the most torque? Prove this mathematically. (Assume we only care about the very start of her very first push, and choose a reference frame for the angle that makes most sense for you).

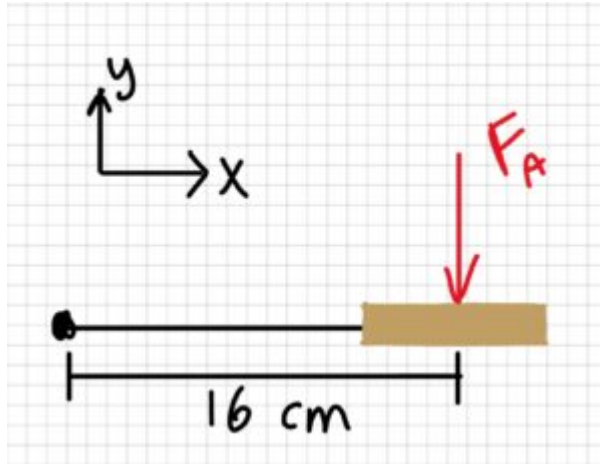
- b) What angle(s) should the bike pedal be at if Jane wants to generate exactly half of the maximum amount of torque?
- c) Is there any position(s) at which the pedal will create zero torque? Where are they and why?



Source: [https://commons.wikimedia.org/wiki/File:Girl\\_on\\_a\\_Bike\\_\(Imagicity\\_116\).jpg](https://commons.wikimedia.org/wiki/File:Girl_on_a_Bike_(Imagicity_116).jpg)



Source: <https://pixabay.com/illustrations/bicycle-cycle-two-wheeler-pedal-3168934/>

**2. Draw****3. Knowns and Unknowns**

Knowns:

$$\begin{aligned} \vec{r} &= \begin{bmatrix} 0.16 \\ 0 \\ 0 \end{bmatrix} \text{ m} \\ \vec{F}_A &= \begin{bmatrix} 0 \\ -100 \\ 0 \end{bmatrix} \text{ N} \end{aligned}$$

Unknowns:

- position of  $r$  for maximum torque
- position of  $r$  for half of maximum torque
- position of  $r$  for zero torque, and why

**4. Approach**

For part a), I will find a general equation for torque based on the given values in terms of  $\theta$ , then analyze the function for its maxima

For part b), I will find the magnitude of 50% of maximum torque and then reverse-engineer the equation to determine what angle(s) the pedal needs to be at to satisfy the equation.

For part c), I will look back at my equation and find when the equation equals zero, then try to understand why given the example problem.

**5. Analysis**

Part a:

$$\begin{aligned} T &= |\vec{F}_A| * |\vec{r}| * \sin \theta \\ T &= (100 \text{ N}) * (0.16 \text{ m}) * \sin \theta \\ T &= 16 \sin \theta \text{ Nm} \end{aligned}$$

$k; k \in \mathbb{Z}$

Thinking about the shape of the sine function in the first period, the maximum occurs at 90 degrees.

You could say algebraically that the maximum is at 90, 450, 810 etc., but these angles all represent the same position on the wheel. Therefore, we will use 90.

Part b:

$$\begin{aligned} T_1 &= \left| F_A \right| \cdot r \cdot \sin \theta \\ T_1 &= (100 \text{ N})(0.16 \text{ m}) \sin 90^\circ \\ T_1 &= 16 \text{ N} \cdot \text{m} \end{aligned}$$

Find 50% of the maximum torque:

$$T_2 = \frac{T_1}{2} \\ \frac{16 \text{ N} \cdot \text{m}}{2} = 8 \text{ N} \cdot \text{m}$$

Rearrange  $T_2$  equation:

$$\begin{aligned} T_2 &= \left| F_A \right| \cdot r \cdot \sin \theta \\ \sin \theta &= \frac{T_2}{\left| F_A \right| \cdot r} \\ \sin \theta &= \frac{8 \text{ N} \cdot \text{m}}{(100 \text{ N})(0.16 \text{ m})} \\ \sin \theta &= 0.5 \quad \theta = 30^\circ, 150^\circ, \text{ etc} \end{aligned}$$

Therefore, Jane could push at 30 from vertical, or 150 from vertical to create half the torque.

\*Interesting to note is that half the angle does not yield half the torque; the angle is 30, not 45. This is because the sine function is non-linear.\*

Part c:

$T = 16 \sin \theta$  tells us that the angles of 0 and 180 will give us zero torque.

This makes sense given that pushing straight down on a stable pendulum will not cause the pendulum to rotate!

Likewise, if you just stand on your pedals, you're providing lots of downward force, but creating zero torque since the crank arm and the direction of the force are parallel (or antiparallel)!

## 6. Review

These answers have the correct units (Nm and degrees) and are within a reasonable order of magnitude based on the given information. See logic/explanations above for more detail.

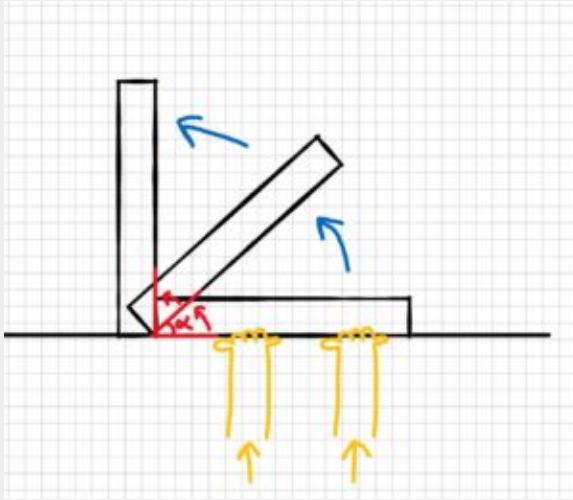
# Example 1.8.5: Torque, submitted by Hamza Ben Driouech

## 1. Problem

A person is pushing on a door with a force of 100 N. The door is at an angle  $\alpha = 45^\circ$  as shown in the sketch below.

a) Calculate the moment when  $r$  is 45 cm and 75 cm.

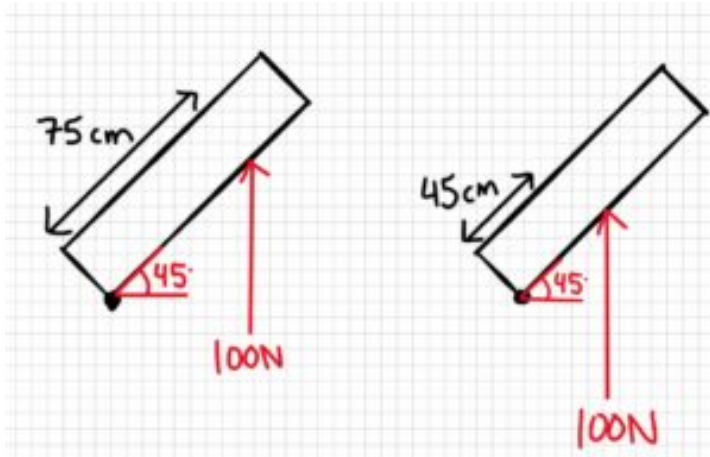
b) At what angle(s) is the moment zero? Explain why.



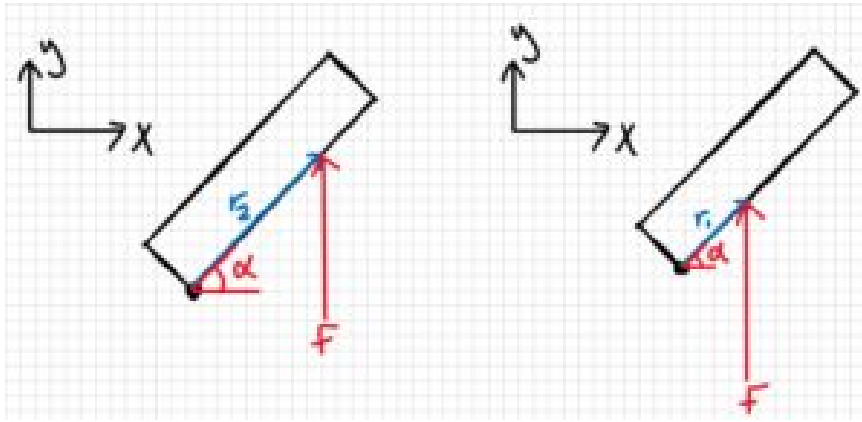
Assumptions: model the force as a single point load acting on the door.

## 2. Draw

Sketch:



Free-body diagram:



### 3. Knowns and Unknowns

Knowns:

- $F = 100 \text{ N}$
- $r_1 = 45 \text{ cm}$
- $r_2 = 75 \text{ cm}$
- $\alpha = 45^\circ$

Unknowns:

- $M_1$ ,  $M_2$ , angle when  $M$  is zero

### 4. Approach

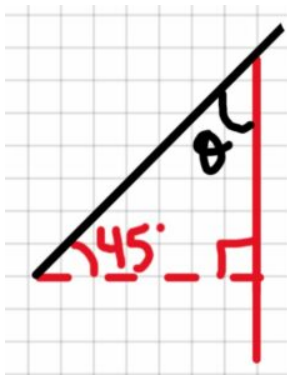
Use equation below.

$$M = |r| \cdot |F| \cdot \sin \theta$$

### 5. Analysis

Part a)

The angle we were given is not technically the one we should use in the moment equation. The angle should be between  $r$  and  $F$ . Therefore, we have to find the new angle.



As shown below, the angle we find is also  $45^\circ$ . Now we can continue and solve for  $M_1$  and  $M_2$ .

$$\theta = 90^\circ - 45^\circ$$

$$\theta = 45^\circ$$

$$M_1 = r_1 \cdot F \cdot \sin(\theta) = 0.45 \text{ m} \cdot 100 \text{ N} \cdot \sin(45^\circ) = 31.82 \text{ Nm} \\ M_2 = r_2 \cdot F \cdot \sin(\theta) = 0.75 \text{ m} \cdot 100 \text{ N} \cdot \sin(45^\circ) = 53.03 \text{ Nm}$$

Part b)

$$M = r \cdot F \cdot \sin(\theta) \text{ if } \sin(\theta) = 0, M = 0 \text{ if } \theta = 0^\circ \text{ or } \theta = 180^\circ, \text{ etc}$$

Answer: the moment is zero when the angle between the force and the moment arm is  $0^\circ$  or  $180^\circ$  (360 would represent the same angle as  $0^\circ$ , as would  $540^\circ$ , etc.)

## 6. Review

It makes sense that the moment is zero when the door is either closed or wide open, because when we apply a force at those positions, no movement of the door is possible.

# Example 1.8.6: Bonus Vector Material, Submitted by Liam Murdock

## 1. Problem

Firstly, George traveled a displacement of  $\mathbf{d}_g = [7 \ 0 \ 8] \text{ m}$  from his car. George's dog named Sparky on the other hand traveled a displacement of  $\mathbf{d}_s = [0 \ 6 \ 6] \text{ m}$  from George's car. Secondly, George called Sparky's name and the dog ran to George's position. It took Sparky four seconds to get there.

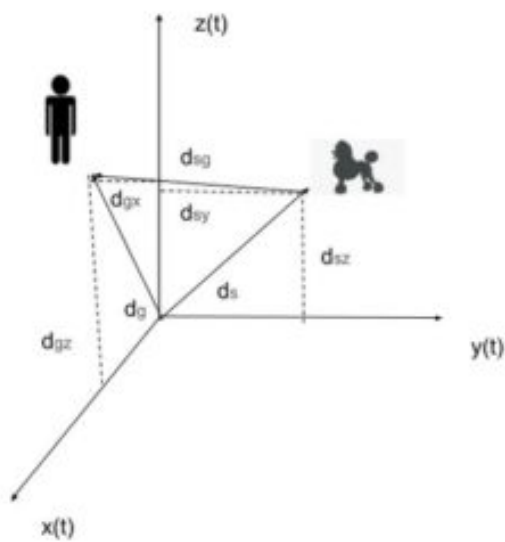
- What is the displacement from George to his dog?
- What is Sparky's velocity? (no need to draw)
- What is Sparky's speed? (no need to draw)





Source: <https://www.piqsels.com/en/public-domain-photo-oekac>

## 2. Draw



## 3. Knowns and Unknowns

Part a:

$$\mathbf{d}_g = [7 \ 0 \ 8] \text{ m} \quad \text{Unknown: } \mathbf{d}_{sg} = ?$$

$$\mathbf{d}_s = [0 \ 6 \ 6] \text{ m}$$

Part b:

$$\mathbf{d}_{sg} = (\text{determined in A}) \quad \text{Unknown: } \mathbf{v}_{sg} = ?$$

$$t = 4 \text{ seconds}$$

Part c:

$$\mathbf{v}_{sg} = (\text{determined in B}) \quad \text{Unknown: } \mathbf{v}_{sg} = ?$$

#### 4. Approach

We are going to use vector operations (both subtraction and division), velocity - displacement relationship, velocity - speed relationship, and pythagoras theorem to solve this problem.

#### 5. Analysis

Part a:

$$\mathbf{d}_{sg} = \mathbf{d}_g - \mathbf{d}_s$$

$$\mathbf{d}_{sg} = [7 \ 0 \ 8] \text{ m} - [0 \ 6 \ 6] \text{ m}$$

$$\mathbf{d}_{sg} = [7-0 \ 0-6 \ 8-6] \text{ m}$$

$$\mathbf{d}_{sg} = [7 \ -6 \ 2] \text{ m}$$

Part b:

$$\mathbf{v}_{sg} = \mathbf{d}_{sg} / t$$

$$\mathbf{v}_{sg} = [7 \ -6 \ 2] \text{ m/s}$$

$$\mathbf{v}_{sg} = [7/4 \ -6/4 \ 2/4] \text{ m/s}$$

$$\mathbf{v}_{sg} = [1.75 \ -1.5 \ 0.5] \text{ m/s}$$

Part c:

$$v_{sg} = |\mathbf{v}_{sg}|$$

$$v_{sg} = (\mathbf{v}_{sgx})^2 + (\mathbf{v}_{sgy})^2 + (\mathbf{v}_{sgz})^2$$

$$v_{sg} = (1.75)^2 + (-1.5)^2 + (0.5)^2$$

$$v_{sg} = 2.36 \text{ m/s}$$

#### 6. Review

Part a:

One way to review the question is to walk through the solution verbally. Our solution shows that for Sparky to get to George, he must walk 7 m in the positive x-direction (almost out of the page), 6 m in the negative y-direction (left), and finally 2 m in the positive z-direction (up).

Firstly, since the dog initially did not go in the x-direction it makes sense Sparky would have to copy George's exact x movement. Secondly, since George did not move in the y - direction, it would make sense that Sparky would just have to retrace his steps and if he initially went 6 m right, he would have to go 6 m left. Thirdly, George and

Sparky both went upwards, but George went 2 m higher with an altitude of 8 m compared to Sparky's 6 m correlating to Spraky having to go positive 2 m in the z - direction to meet Geroge.

Therefore, since all the movements make sense for Sparky to meet George (using logic), the answer is proven to be right.

Part b and c:

Since B and C correlate to the same magnitude they can be reviewed together. From a quick search, an average dog tops out at a speed of 19 miles per hour. We can convert this to SI units:

$$\left(\frac{19 \text{ miles}}{1 \text{ hr}}\right)\left(\frac{1 \text{ km}}{0.621371 \text{ miles}}\right)\left(\frac{1000 \text{ m}}{1 \text{ km}}\right)\left(\frac{1 \text{ hr}}{3600}\right)=8.49 \text{ m/s}$$

The top speed of an average dog is 8.49 m/s. So 2.36m/s is approximately a quarter of the top speed of an average dog. Sparky probably was not sprinting at full speed and he could be a slower dog breed, making 2.36m/s a reasonable answer.

# CHAPTER 2: PARTICLES

In this chapter, we analyze our first static bodies (motion where acceleration = 0), treating them as particles. The sections in this chapter include:

- [2.1 Particle & Rigid Body](#) - the difference between particles and rigid bodies
- [2.2 Free Body Diagrams for Particles](#) - learning how to model forces and motion
- [2.3 Equilibrium Equations for Particles](#) - analyzing static bodies
- [2.4. Examples](#) - problems submitted by other students.

Very simply, here are the important equations for this section (the [Ch 1 equations](#) might be helpful too)

## Ch2 Equations

*Particle* (non-deforming mass)

*Rigid Body* (non-deforming  
mass & shape)

### *Particle Free-Body Diagram Tips*

- Coordinate frame
- Replace surfaces with forces (floor, hand, objects)
- Forces in the correct direction
- Unique (different) names

### *Particle Equilibrium Equations*

$$\sum \vec{F} = 0 \quad \begin{aligned} \sum F_x &= 0 \\ \sum F_y &= 0 \\ \sum F_z &= 0 \end{aligned}$$

## 2.1 Particle & Rigid Body

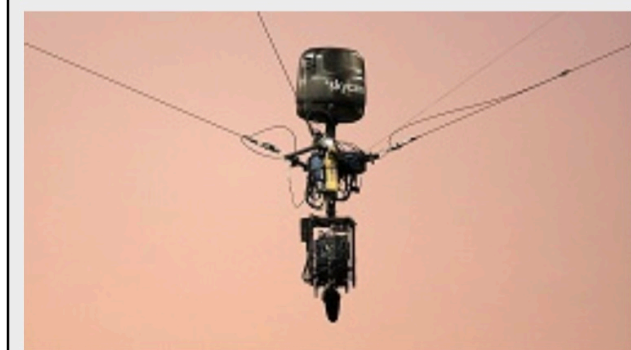
This may seem like a simple concept, but it is important to know the difference between particles and rigid bodies, because it will change the type of analysis you perform.

**Particles** are bodies where all the mass is concentrated at a single point in space. Particle analysis will only have to take into account the forces acting on the body and translational motion because rotation is not considered for particles.

**Rigid bodies** on the other hand have mass that is distributed throughout a finite volume. Rigid body analysis is more complex and also has to take into account moments and rotational motions. In actuality, no bodies are truly particles, but some bodies can be approximated as particles to simplify analysis. Bodies are often assumed to be particles if the rotational motions are negligible when compared to the translational motions, or in systems where there is no moment exerted on the body such as a concurrent force system.



The rotation of this comet and the moments exerted on the comet are unimportant in modeling its trajectory through space, therefore we would treat it as a particle. Public Domain image by Buddy Nath.



The gravitational forces and the tension forces on the skycam all act through a single point, making this a concurrent force system that can be analyzed as a particle. Image by Despeaux CC-BY-SA 3.0.



Rotation and moments will be key to the analysis of the crowbar in this system, therefore the crowbar needs to be analyzed as an extended body. Public Domain image by Pearson Scott Foresman.

Source: Engineering Mechanics, Jacob Moore, et al.

<http://www.oercommons.org/courses/mechanics-map-open-mechanics-textbook/view>

Particles are typically part of a larger scale, such as a sky diver falling through the sky, or a football flying through the air. **Rigid body analyses are required when the length or size of the object must be considered**, such as if you need to calculate the torque from turning a bolt with a wrench, or if there is rotation, such as the bolt that is being turned.

One way to think of it is that *particles have mass*, whereas *rigid bodies have mass and shape*. We make an assumption that neither particles nor rigid bodies deform (change shape). Note: we say particles don't deform even though we are already assuming that the shape of particles is negligible.

In baseball, if you want to consider how far the ball travels, that would be a particle analysis because the speed is much greater than the size of the ball. A rigid body analysis would be how the bat swings to hit the ball, because the length of the bat would change how far the ball travels. A rigid body analysis could be to calculate the spin on the ball as it flies through the air (if you focus on how it is rotating).





Source: <https://i0.hippopx.com/photos/613/24/1019/softball-batter-girl-game-preview.jpg>

You would have done particle analyses in your high school physics classes. Starting in chapter 3, we'll expand on these concepts to include rigid bodies and bring shape and size into the problem.

#### Key Takeaways

*Basically:* Particles have non-deforming mass & rigid bodies have non-deforming mass with shape & size. Rigid body analyses are required when length or size of the object much be considered, including rotation and torque. Particle analyses are for a grander scale where the object is small in comparison to the distance or speed.

*Application:* A particle analysis would be an airplane as it flies at a high speed through the air. A rigid body analysis would be if you analyze how the plane is rotating in order to turn, or to consider the size of the wheels as it is taxiing on the runway or the size of the wings to keep it in the air.

*Looking ahead:* Chapter 2 concerns particle analyses and Chapters 3 – 7 focus on rigid body analyses.

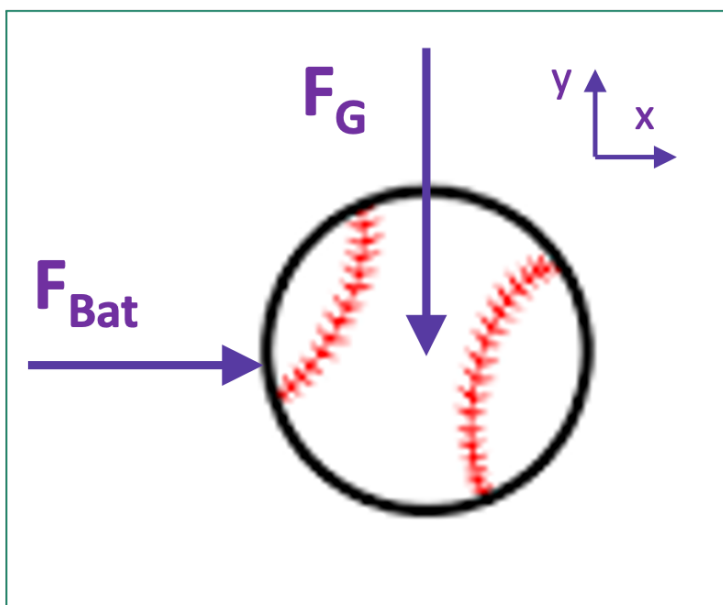
## 2.2 Free Body Diagrams for Particles

A free-body diagram (FBD) helps you to simplify a complicated problem. The first thing to remember is the object should always be *free* which means, floating in space. You represent the floor or other surfaces with forces. You might have done these particle free body diagrams in your high school physics class, where all the forces act at the centre of the object. (This will be different for rigid bodies).

To draw a free-body diagram remember four points:

- Add coordinate frame (which way is positive  $x$  and positive  $y$ ?)
- Replace surfaces with forces (floor, hand, and objects touching it become arrows)
- Point forces in the correct direction (the head of the arrow points to where the force acts.  $F_G$  acts down)
- Use unique (different) names (be sure to name each force with a different name).

For a baseball being hit by a bat (and neglecting air), the force of gravity acts at the center, the force of the bat acts on the outside. Notice in the figure the names  $F_{\text{Bat}}$  and  $F_G$  are different in the figure below. Also - you can understand what they represent quickly. Also see the coordinate frame? You'll be adding these in your sleep by the end of this class.



As you draw a free body diagram, there are a couple of things you need to keep in mind:

- (1) Include only those forces acting ON the object whose free body diagram you are drawing. Any force exerted BY the object on some other object belongs on the free body diagram of the other object.



(2) All forces are contact forces and every force has an agent. The agent is “that which is exerting the force.” In other words, the agent is the life form or thing that is doing the pushing or pulling on the object. No agent can exert a force on an object without being in contact with the object.

We are going to introduce the various kinds of forces by means of examples. Here is the first example:

A rock is thrown up into the air by a person. Draw the free body diagram of the rock while it is up in the air. (Your free body diagram is applicable for any time after the rock leaves the thrower’s hand, until the last instant before the rock makes contact with whatever it is destined to hit.) Neglect any forces that might be exerted on the rock by the air.

If you see the rock flying through the air, it may very well look to you like there is nothing touching the rock. But the earth’s gravitational field is everywhere in the vicinity of the earth. It can’t be blocked. It can’t be shielded. It is in the air, in the water, even in the dirt. It is in direct contact with everything in the vicinity of the earth. It exerts a force on every object near the surface of the earth. We call that force the gravitational force. You have already studied the gravitational force. We give a brief synopsis of it here.

### The Gravitational Force Exerted on Objects Near the Surface of the Earth.

Because it has mass, the earth has a gravitational field. The gravitational field is a force-per-mass field. It is invisible. It is not matter. It is an infinite set of force-per-mass vectors, one at every point in space in the vicinity of the surface of the earth. Each force per-mass vector is directed downward, toward the center of the earth and, near the surface of the earth, has a magnitude of 9.81 N/kg. The effect of the earth’s gravitational field is to exert a force on any object that is in the earth’s gravitational field. The force is called the gravitational force and is equal to the product of the mass of the object and the earth’s gravitational field vector:  $F_g = mg$ . Where  $g = 9.81$  N/kg is the magnitude of the earth’s gravitational field vector. The direction of the near-earth’s-surface gravitational force is downward, toward the center of the earth.

Here is the free body diagram and the corresponding table of forces:

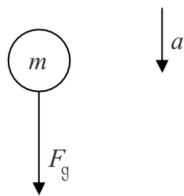


Table of Forces			
<i>Symbol = ?</i>	<i>Name</i>	<i>Agent</i>	<i>Victim</i>
$F_g = mg$	Gravitational Force	The Earth’s Gravitational Field	The Rock

(1) The only thing touching the object while it is up in the air (neglecting the air itself) is the earth’s gravitational field. So there is only one force on the object, namely the gravitational force. The arrow representing the force vector is drawn so that the tail of the arrow is touching the object, and the arrow extends away from the object in the direction of the force.

(2) Unless otherwise stipulated, label the diagram yourself however it makes most sense. Always draw a coordinate frame (Usually x is upwards, and y extends to the right).

(3) There is no velocity information on a free body diagram

(4) There is no force of the hand acting on the object because, at the instant in question, the hand is no longer touching the object. When you draw a free body diagram, only forces that are acting on the object at the instant depicted in the diagram are included. The acceleration of the object depends only on the currently-acting forces on the object. The force of the hand is of historical interest only.

(5) Regarding the table of forces:

- Make sure that for any free body diagram you draw, you are capable of making a complete table of forces. You are not required to provide a table of forces with every free body diagram you draw, but you should expect to be called upon to create a table of forces more than once.
- In the table of forces, the agent is the life form or thing that is exerting the force and the victim is the object on which the force is being exerted. Make sure that, in every case, the victim is the object for which the free body diagram is being drawn.
- In the case at hand, there is only one force so there is only one entry in the table of forces.
- For any object near the surface of the earth, the agent of the gravitational force is the earth's gravitational field. It is okay to abbreviate that to "Earth" because the gravitational field of the earth can be considered to be an invisible part of the earth, but it is NOT okay to call it "gravity." Gravity is a subject heading corresponding to the kind of force the gravitational force is, gravity is not an agent

A ball of mass  $m$  hangs at rest, suspended by a string. Draw the free body diagram for the ball, and create the corresponding table of forces.

To do this problem, you need the following information about strings:

**The Force Exerted by a Taut String on an Object to Which it is Affixed** (This also applies to ropes, cables, chains, and the like.)

The force exerted by a string, on an object to which it is attached, is always directed away from the object, along the length of the string. Note that the force in question is exerted by the string, not for instance, by some person pulling on the other end of the string. The force exerted by a string on an object is referred to as a "tension force" and its magnitude is conventionally represented by the symbol  $F_T$ .

Note: There is no formula to tell you what the tension force is. If it is not given, the only way to get it is to use Newton's 2nd Law.

Here is the free body diagram of the ball, and the corresponding table of forces:

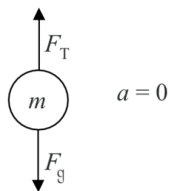


Table of Forces			
Symbol=?	Name	Agent	Victim
$F_T$	Tension Force	The String	The Ball
$F_g = mg$	Gravitational Force	The Earth's Gravitational Field	The Ball

*There is no "force of motion" acting on an object. Once you have the force or forces exerted on the object by everything that is touching the object, you have all the forces. Do not add a "force of*

*motion” to your free body diagram. It is especially tempting to add this force when there are no actual forces in the direction in which an object is going. Keep in mind, however, that an object does not need a force on it to keep going in the direction in which it is going; moving along at a constant velocity is what an object does when there is no net force on it.*

Source: Calculus-Based Physics 1, Jeffery W. Schnick.

<https://openlibrary.ecampusontario.ca/catalogue/item/?id=ce74a181-ccde-491c-848d-05489ed182e7> page 86

### Key Takeaways

*Basically:* Free-body diagrams (FBDs) give you a way to model complicated problem in a simple way. All exterior forces are modeled with an arrow.

*Application:* A baseball can be modeled using a FBD to show how the bat and gravity affect the ball.

*Looking ahead:* You’ll use a FBD in every step 2 in nearly every homework problem. These are especially helpful with Equilibrium Equations in the next section.

Original image of baseball from: <https://openclipart.org/detail/258473/baseball-refixed> Annotations added by the author

## 2.3 Equilibrium Equations for Particles

For a particle in static equilibrium, Newton's 2nd law can be adapted for  $\vec{a} = 0$  and componentized in x y and z:

$$\sum \vec{F} = m \vec{a}$$

$$\sum \vec{F} = 0$$

$$\sum F_x = 0 \quad \quad \sum F_y = 0 \quad \quad \sum F_z = 0$$

Notice that the left side of the equation says 'sum of the forces' which means add up all the forces in that direction. In statics, they will all cancel out. If you aren't sure if something is in static motion, sum the forces and see if they equal 0.

### Static Equilibrium:

Objects in static equilibrium are objects that are not accelerating (either linear acceleration or angular acceleration). These objects may be stationary, or they may have a constant velocity.

Newton's Second Law states that the force exerted on an object is equal to the mass of the object times the acceleration it experiences. Therefore, if we know that the acceleration of an object is equal to zero, then we can assume that the sum of all forces acting on the object is zero. Individual forces acting on the object, represented by force vectors, may not have zero magnitude but the sum of all the force vectors will always be equal to zero for objects in equilibrium.

The equations used when dealing with particles in equilibrium are:

$$\sum \vec{F} = 0$$

Which leads to:

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$

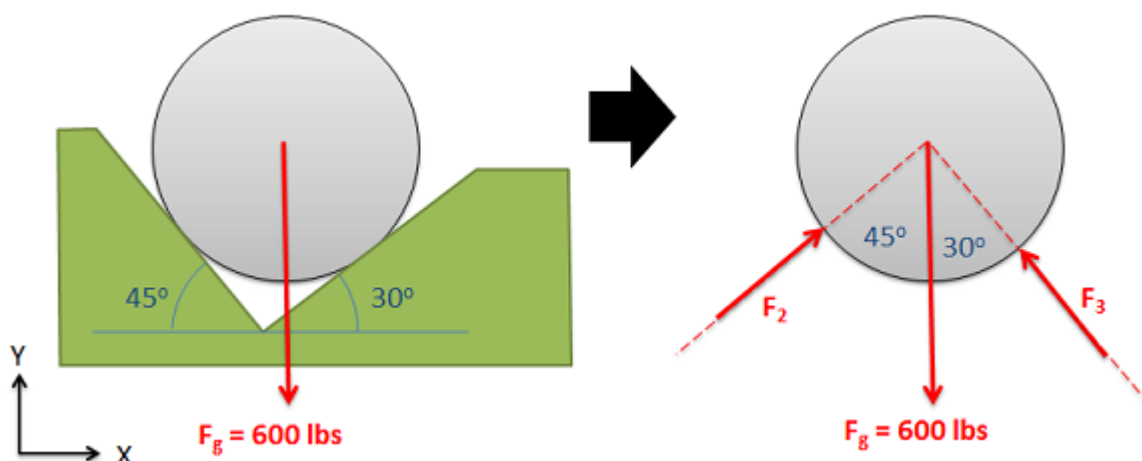
Since it is a particle, there are no moments involved like there is when it comes to rigid bodies.

Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/2\\_equilibrium\\_concurrent/2-1\\_static\\_equilibrium/staticequilibrium.html](http://mechanicsmap.psu.edu/websites/2_equilibrium_concurrent/2-1_static_equilibrium/staticequilibrium.html)

## Finding the Equilibrium Equations:

The first step in finding the equilibrium equations is to **draw a free body diagram** of the body being analyzed. This diagram should show all the known and unknown force vectors acting on the body. In the free body diagram, provide values for any of the known magnitudes or directions for the force vectors and provide variable names for any unknowns (either magnitudes or directions).



The first step in equilibrium analysis is drawing a free body diagram. This is done by removing everything but the body and drawing in all forces acting on the body. It is also useful to label all forces, key dimensions, and angles.

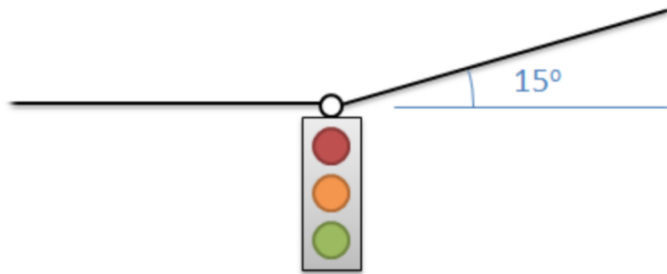
Next you will need to choose the  $x$ ,  $y$ , and  $z$  axes. These axes do need to be perpendicular to one another, but they do not necessarily have to be horizontal or vertical. If you choose coordinate axes that line up with some of your force vectors you will simplify later analysis.

Once you have chosen axes, you need to break down all of the force vectors into components along the  $x$ ,  $y$  and  $z$  directions (see the vectors page in Appendix 1 if you need more guidance on this). Your first equation will be the sum of the magnitudes of the components in the  $x$  direction being equal to zero, the second equation will be the sum of the magnitudes of the components in the  $y$  direction being equal to zero, and the third (if you have a 3D problem) will be the sum of the magnitudes in the  $z$  direction being equal to zero. Collectively these are known as the **equilibrium equations**.

Once you have your equilibrium equations, you can solve them for unknowns using algebra. The number of unknowns that you will be able to solve for will be the number of equilibrium equations that you have. In instances where you have more unknowns than equations, the problem is known as a **statically indeterminate problem** and you will need additional information to solve for the given unknowns.

Example:

A 6 kg traffic light is supported by two cables as shown below. Find the tension in each of the cables supporting the traffic light.

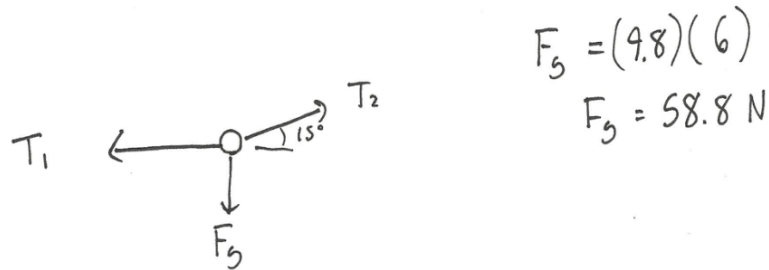


$$F_g = (9.8)(6) = 58.8 \text{ N}$$

$$\sum F_x = -T_1 + T_2 \cos(15^\circ) = 0$$

$$F_y = T_2 \sin(15^\circ) - 58.8 = 0 \quad T_2 = \frac{58.8}{\sin(15^\circ)} = 227.2 \text{ N}$$

$$T_1 + 227.2 \cos(15^\circ) = 0 \quad T_1 = 227.2 \cos(15^\circ) = 219.4 \text{ N} \quad T_2 = 227.2 \text{ N}$$



$$\sum F_x = -T_1 + T_2 \cos(15^\circ) = 0$$

$$\sum F_y = T_2 \sin(15^\circ) - 58.8 = 0$$

$$T_2 = \frac{58.8}{\sin(15^\circ)} = 227.2 \text{ N}$$

$$-T_1 + 227.2 \cos(15^\circ) = 0$$

$$T_1 = 227.2 \cos(15^\circ) = 219.4 \text{ N}$$

∴

$$T_1 = 219.4 \text{ N} \quad T_2 = 227.2 \text{ N}$$

Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/2\\_equilibrium\\_concurrent/2-5\\_equilibrium\\_equations\\_particle/equilibrium-equationsparticle.html](http://mechanicsmap.psu.edu/websites/2_equilibrium_concurrent/2-5_equilibrium_equations_particle/equilibrium-equationsparticle.html) Many more examples are available at this site.

See additional examples: [Example 1](#), [Example 5](#), [Example 8](#)

### Key Takeaways

*Basically:* Equilibrium equations allow you to separate the forces and motion into each axis.

*Application:* A heavy object is lifted using a rope and pulley. Based on the forces, is it in static equilibrium?  
(Answer: if sum of the forces in each direction equal zero, then yes!)

*Looking ahead:* We will next apply equilibrium equations to rigid bodies, allowing use to solve more complex problems.

## 2.4. Examples

Here are examples from Chapter 2 to help you understand these concepts better. These were taken from the real world and supplied by FSDE students in Summer 2021. If you'd like to submit your own examples, please send them to the author [eosgood@upei.ca](mailto:eosgood@upei.ca).

No examples submitted from students, yet. In the mean time, here are examples ( [Example 1](#), [Example 5](#), [Example 8](#)) from Engineering Mechanics, Jacob Moore, et al. [http://mechanicsmap.psu.edu/websites/2\\_equilibrium\\_concurrent/2-5\\_equilibrium\\_equations\\_particle/equilibrium-equations-particle.html](http://mechanicsmap.psu.edu/websites/2_equilibrium_concurrent/2-5_equilibrium_equations_particle/equilibrium-equations-particle.html)



# CHAPTER 3: RIGID BODY BASICS

In this chapter, you will learn some fundamental tools for rigid bodies, what I call the *rigid body basics*. Recall, rigid bodies have mass and a particular shape or size. Here are the sections in this Chapter:

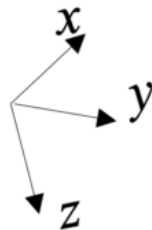
- 3.1 Right Hand Rule - a way to help you make accurate coordinate frames
- 3.2 Couples - rotational motion created from two forces
- 3.3 Distributed Loads - a way to express a force over a certain area
- 3.4 Reactions & Supports - how to model the constraints that keep an object in place
- 3.5 Indeterminate Loads - how to determine if there are too many forces
- 3.6 Examples - examples from your peers

Here are the important equations for this chapter.

## Ch3 Equations

Right hand rule

$$\hat{i} \times \hat{j} = \hat{k}$$



$$\text{Intensity (w)} = F / d$$

$$w [=] \text{ N/m or lb/ft}$$

Resultant Force ( $F_R$ ) = Area of arrows

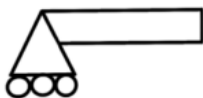
$$F_R = w * L$$

$$F_{eq} = \int_{x_{min}}^{x_{max}} F(x) dx$$

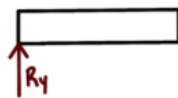
Location of Resultant Force ( $\bar{x}$ )

$$\bar{x} = \frac{\sum Fx}{\sum F} \quad \bar{x} = \frac{\int x w(x) dx}{\int w(x) dx}$$

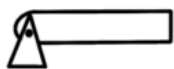
Roller



Reaction Forces



Pinned



Fixed / cantilever



Shape	Equivalent Point Load	Force Equation	Location
		$F = \frac{1}{2}bh = \frac{1}{2}dw_{max}$	$\bar{x} = d/3$
		$F = bh = wd$	$\bar{x} = d/2$
		$F = \frac{1}{2}b(h_1 + h_2)$ $= \frac{1}{2}d(w_1 + w_2)$	$\bar{x} = \frac{d(w + 2w_2)}{3(w_1 + w_2)}$

## 3.1 Right Hand Rule

Before we can analyze rigid bodies, we need to learn a little trick to help us with the cross product called the 'right-hand rule'. We use the right-hand rule when we have two of the axes and need to find the direction of the third.

This is called a right-orthogonal system. The '*orthogonal*' part means that the three axes are all perpendicular to each other, and the '*right*' part means that

$\underline{\hat{i}} \times \underline{\hat{j}} = \underline{\hat{k}}$ , hence the right hand rule. Remember these from section 1.5?

- $\underline{\hat{i}} \times \underline{\hat{j}} = \underline{\hat{k}}$
- $\underline{\hat{j}} \times \underline{\hat{k}} = \underline{\hat{i}}$
- $\underline{\hat{k}} \times \underline{\hat{i}} = \underline{\hat{j}}$
- $\underline{\hat{j}} \times \underline{\hat{i}} = -\underline{\hat{k}}$
- $\underline{\hat{k}} \times \underline{\hat{j}} = -\underline{\hat{i}}$
- $\underline{\hat{i}} \times \underline{\hat{k}} = -\underline{\hat{j}}$

The opposite of the right-orthogonal system is the left-orthogonal system where  $\underline{\hat{i}} \times \underline{\hat{j}} = -\underline{\hat{k}}$ . We don't use that one!

There are two ways to do the right hand rule, and they take practice to conceptually understand, but this will make solving problems much quicker. You're going to use your fingers and thumb to represent the x, y, and z axes.

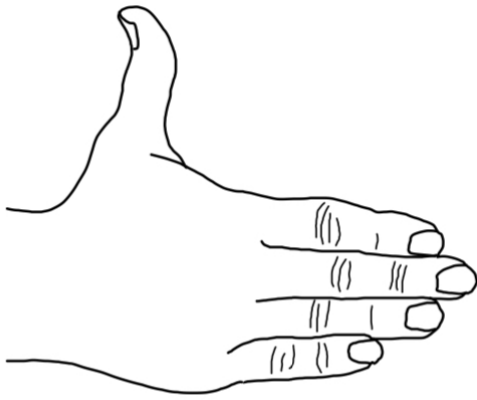
### 3.1.1 The Whole-Hand Method

In the following description,  $A \times B = C$ , so for the coordinate frame,  $X \times Y = Z$

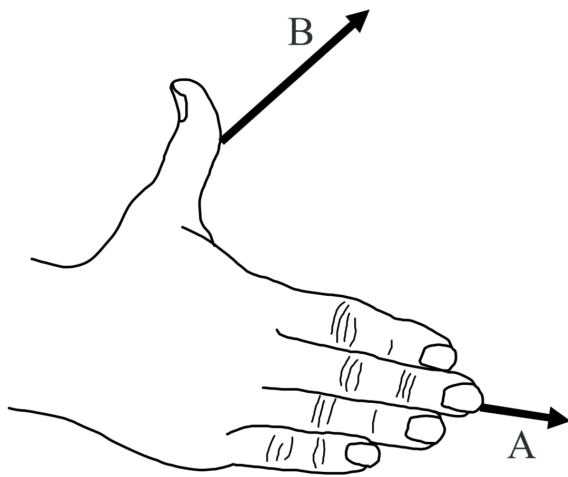
( $\underline{\hat{i}} \times \underline{\hat{j}} = \underline{\hat{k}}$ ). Your fingers go in the direction of X, then you bend them 90 degrees to point towards Y, and your thumb is in the direction of Z.

The direction of the cross product vector  $A \times B$  is given by the right-hand rule for the cross product of two vectors. To apply this right-hand rule, extend the fingers of your right hand so that they are pointing directly away

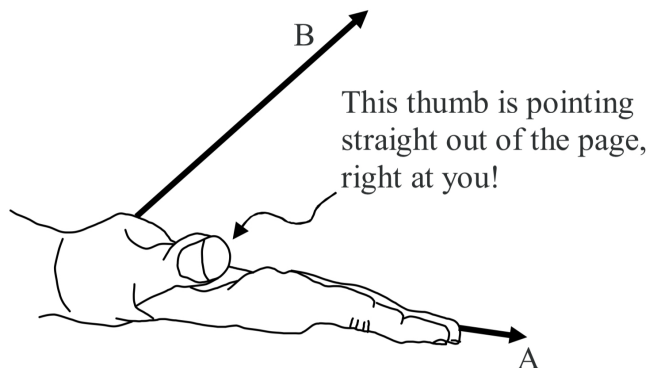
from your right elbow. Extend your thumb so that it is at right angles to your fingers.



Keeping your fingers aligned with your forearm, point your fingers in the direction of the first vector (the one that appears before the “ $\times$ ” in the mathematical expression for the cross product; e.g. the  $A$  in  $A \times B$  ).



Now rotate your hand, as necessary, about an imaginary axis extending along your forearm and along your middle finger, until your hand is oriented such that, if you were to close your fingers, they would point in the direction of the second vector.



Your thumb is now pointing in the direction of the cross product vector.  $C = A \times B$ . The cross product vector  $C$  is always perpendicular to both of the vectors that are in the cross product (the  $A$  and the  $B$  in the case at hand). Hence, if you draw them so that both of the vectors that are in the cross product are in the plane of the page, the

cross product vector will always be perpendicular to the page, either straight into the page, or straight out of the page. In the case at hand, it is straight out of the page.

When we use the cross product to calculate the torque due to a force  $F$  whose point of application has a position vector  $r$ , relative to the point about which we are calculating the torque, we get an axial torque vector  $\tau$ . To determine the sense of rotation that such a torque vector would correspond to, about the axis defined by the torque vector itself, we use The Right Hand Rule For Something Curly Something Straight. Note that we are calculating the torque with respect to a point rather than an axis—the axis about which the torque acts, comes out in the answer.

Source: Jeffrey W. Schnick

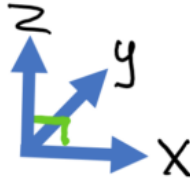
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=ce74a181-ccde-491c-848d-05489ed182e7> pages 135-137

The hardest part of right-hand rule is imagining the different axes and envisioning how they are perpendicular to each other.

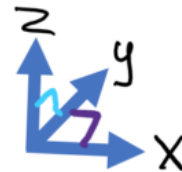
We use a right angle symbol to show when two axes are perpendicular:



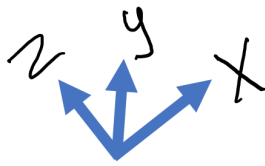
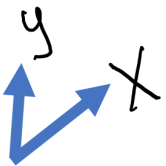
See how x and z are perpendicular (using a green right angle symbol):



Then see how x & y are perpendicular (purple) and y & z are perpendicular (blue):

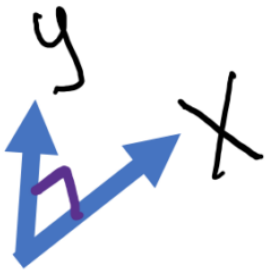


Try this one in 2d and 3d. Imagine (or draw) the right-angle symbols (Answer will be in a few steps)



## Example 1:

Using this x and y, let's use the right-hand rule to find the direction of z.



Here are steps you can follow:

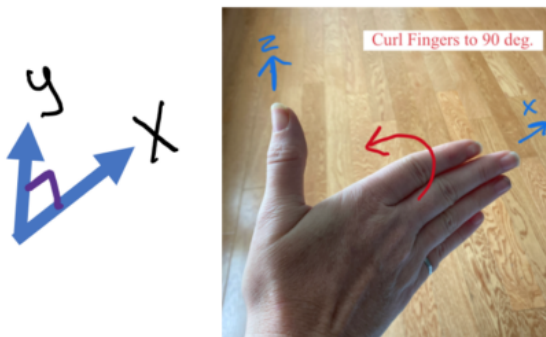
1. Make sure you use your right hand & not your left hand. This is easy to forget.



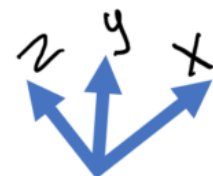
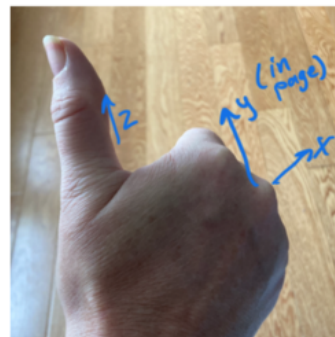
2. Point your fingers towards the x axis



3. Curl your fingers then towards the y axis (into the page).



4. So according the right-hand rule, the z direction is up and to the left.

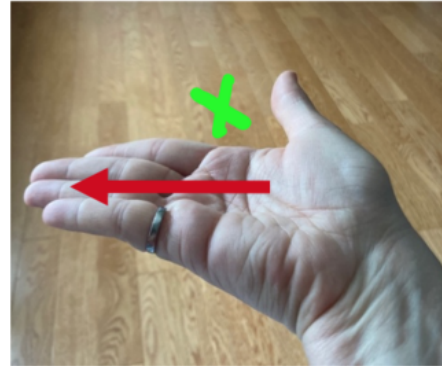
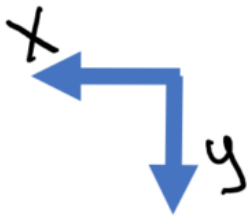


## Example 2:

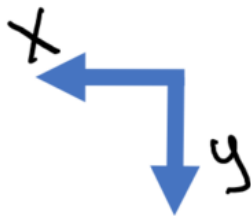
Sometimes you will need to flip your hand 180 degrees to find which way lets you point your fingers in

the y direction, for example:

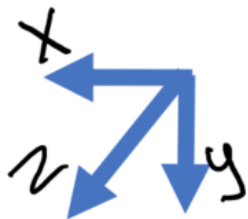
1. Put your fingers in the direction of x. Both of these have x towards the left.



2. Now bend your fingers 90 degrees towards y. You may have to flip your hand over.  
(Only one of these is correct):



3. Your thumb shows the direction of z.  
(Only one is correct):

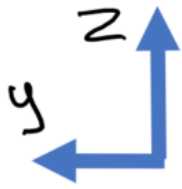


Can you 'see' how the coordinate frames are orthogonal (perpendicular)?

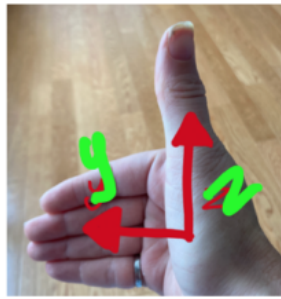


## Example 3:

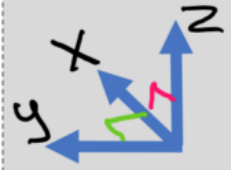
It's important for you to be able to envision how the axes are perpendicular. Now practice using the right hand rule if you are trying to find x.

**To find x:**

1. Put your thumb towards z.
2. Put your fingers bent at an angle of 90 degrees in the direction of y.
3. To find x, simply uncurl your fingers.

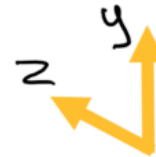
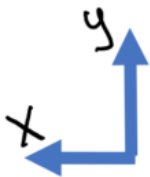


**ANSWER:**  
**X goes into the page.**  
 Can you 'see' how the coordinate frames are orthogonal?



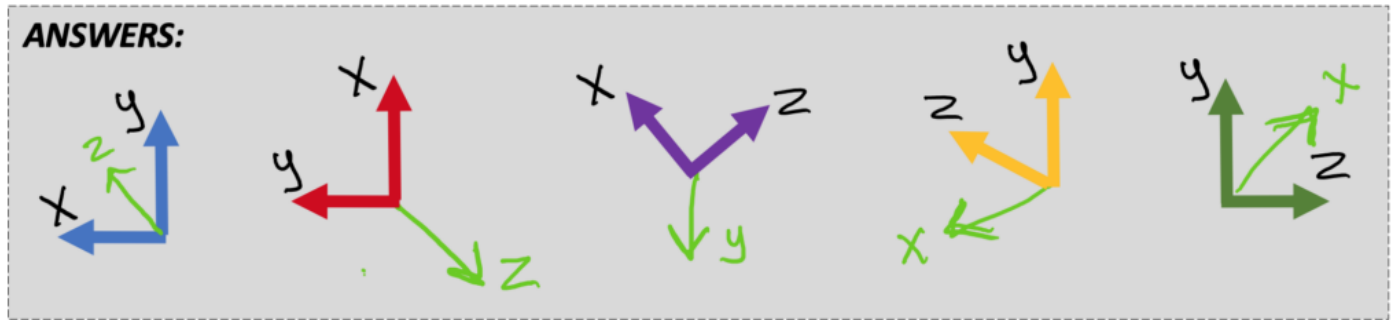
## Your Turn!

Keep going with these examples. The rules stay the same: thumb towards z, curled fingers towards y, extended fingers towards x. Find the missing axis:



Did you do it?

Here are the answers:



### 3.1.2 Right Hand Rule and Torque

The third way to calculate torque, as was alluded to in Section 1.6 is to use the right hand rule to identify the axis of rotation. The first way (the scalar method) uses  $|\mathbf{M}| = |\mathbf{r}| |\mathbf{F}| \sin \Theta$ , and often the angle between the position vector and force is 90 degrees. The vector method is for more complicated situations and uses the cross product  $\mathbf{r} \times \mathbf{F} = \mathbf{M}$ . The third method finds the scalar value separately, then uses the right hand rule to find the direction (positive or negative along the third axis).

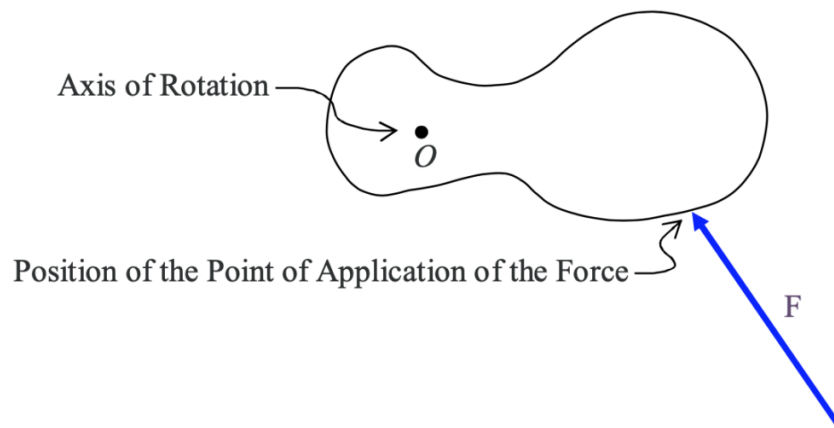
- Point your fingers in the direction of *the perpendicular part* of position vector  $\mathbf{r}$  (as you would for x)
- Curl them towards the direction of the Force vector  $\mathbf{F}$  (as you would for y)
- Your thumb is in the direction of the moment  $\mathbf{M}$  that results from the force (as for z)

The following will help you understand what is meant by: *the perpendicular part* of position vector:

The torque  $\tau$  can be expressed as the cross product of the position vector  $\mathbf{r}$  for the point of application of the force, and the force vector  $\mathbf{F}$  itself:  $\mathbf{r} \times \mathbf{F} = \mathbf{M}$

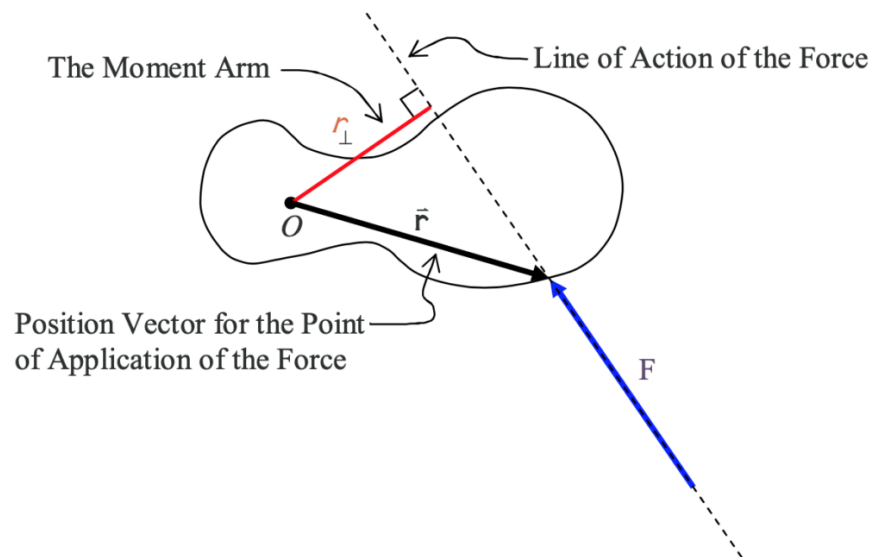
Before we begin our mathematical discussion of what we mean by the cross product, a few words about the vector  $\mathbf{r}$  are in order. It is important for you to be able to distinguish between the position vector  $\mathbf{r}$  for the force, and the moment arm, so we present them below in one and the same diagram. We use the same example that we have





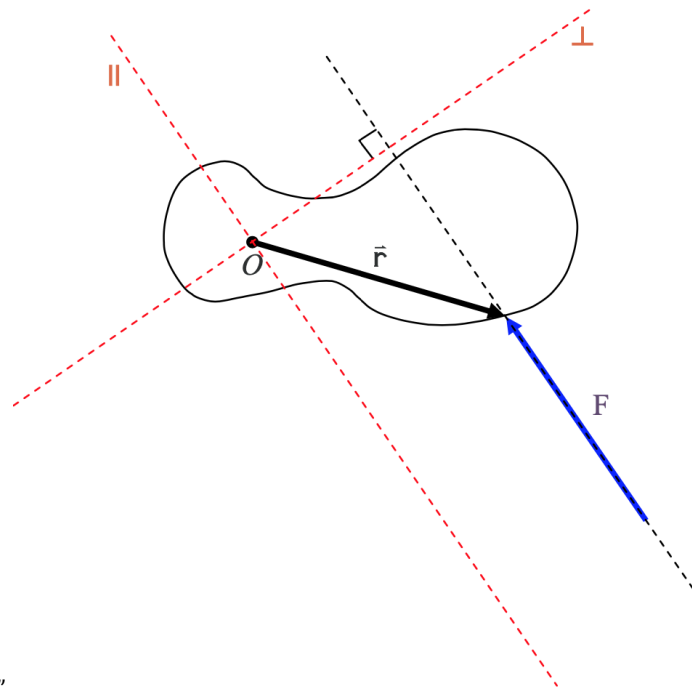
used before:

in which we are looking directly along the axis of rotation (so it looks like a dot) and the force lies in a plane perpendicular to that axis of rotation. We use the diagrammatic convention that, the point at which the force is applied to the rigid body is the point at which one end of the arrow in the diagram touches the rigid body. Now we add the line of action of the force and the moment arm  $r_{\perp}$  to the diagram, as well as the position vector  $\vec{r}$  of the



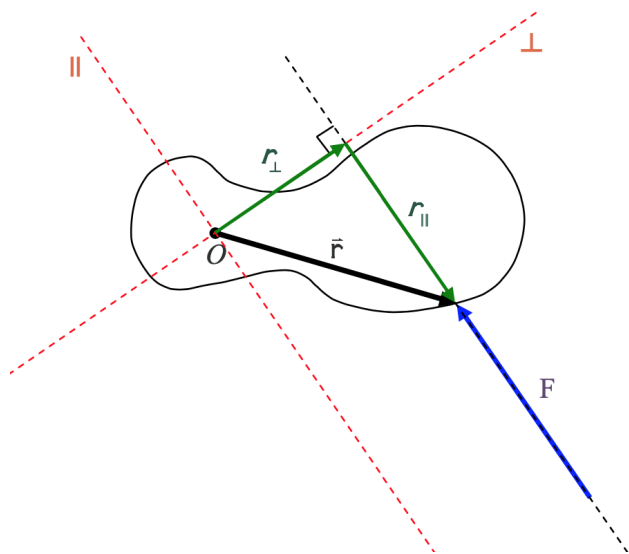
point of application of the force.

The moment arm can actually be defined in terms of the position vector for the point of application of the force. Consider a tilted x-y coordinate system, having an origin on the axis of rotation, with one axis parallel to the line of action of the force and one axis perpendicular to the line of action of the force. We label the x axis  $\perp$  for



“perpendicular” and the y axis || for “parallel”.

Now we break up the position vector  $\vec{r}$  into its component vectors along the  $\perp$  (perpendicular) and  $\parallel$  (parallel) axes.



From the diagram it is clear that the moment arm  $r$  is just the magnitude of the component  $\perp$  vector, in the perpendicular-to-the-force direction, of the position vector of the point of application of the force.

Source: Calculus Based Physics, Jeffrey W. Schnick, <https://openlibrary.ecampusontario.ca/catalogue/item/?id=ce74a181-ccde-491c-848d-05489ed182e7> pages 132-137

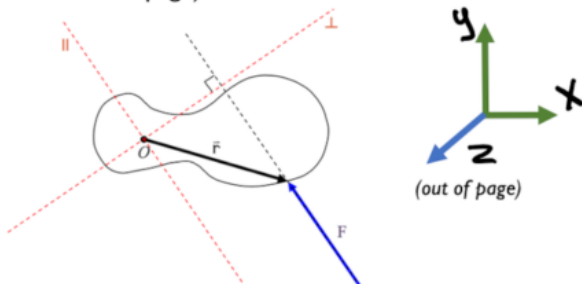
You use the right hand rule twice during this method to find the vector. First to determine the coordinate frame and again to see which the direction the torque is aligned. Then you multiply by the magnitude of the perpendicular portion of the position vector ( $r_{\perp}$  or the “moment arm”) and the magnitude of the force vector. ):

$$|\underline{M}| = +/- |\underline{r}_{\perp}| |\underline{F}|$$

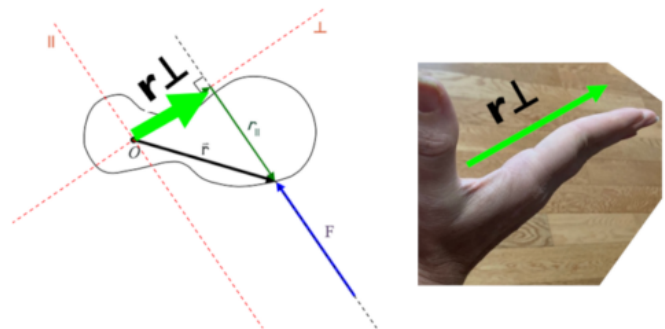
\* though it's not always the  $\hat{k}$  direction, it could be  $\hat{i}$  or  $\hat{j}$  as well. It depends how you define your coordinate frame.

## Example 4:

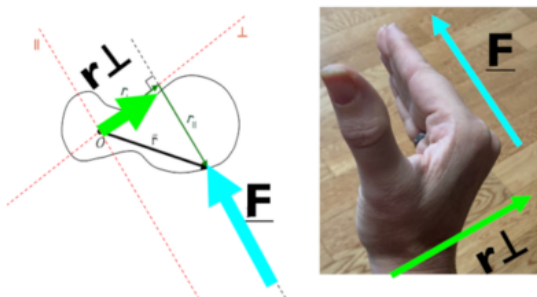
1. Make a coordinate frame with the x axis and y-axis 90 degrees from each other, often x is to the right and y is up. Use the right hand rule to determine the direction of the z axis (whether it's out of the page or into the page).



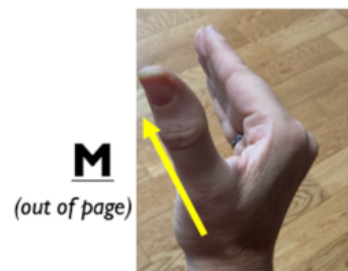
2. Determine where the perpendicular part of the position vector ( $r_{\perp}$ ) is acting. Point your fingers in the direction of ( $r_{\perp}$  up to the right).



3. Curl your fingers in the direction of the Force vector (up to the left).



4. Your thumb should point in either the positive or negative z direction. If it matches the z direction axis, it's positive. If not, it's negative.



Both are  
out of the page  
(though it's hard to  
envision on your  
screen),  
so it's **positive**

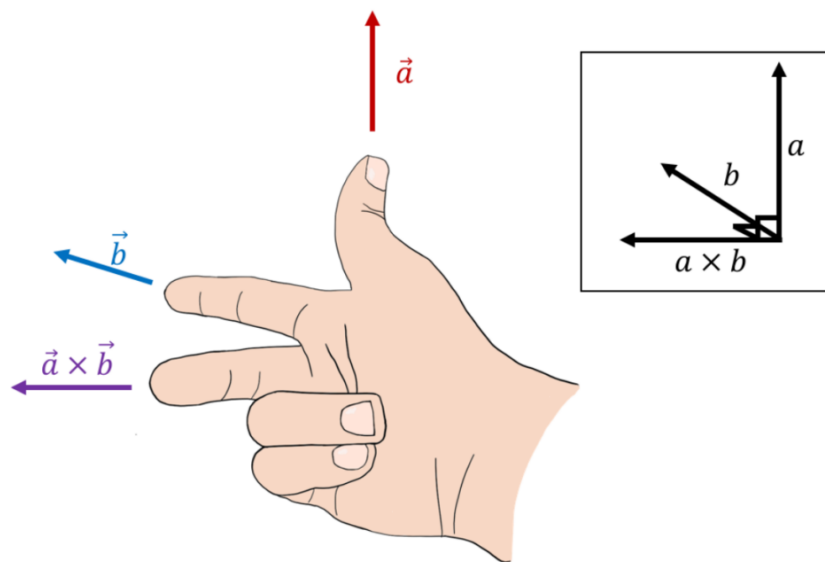
5. The moment equation is the magnitude of the perpendicular portion of the position vector ( $r_{\perp}$ ) times the magnitude of the force ( $|F|$ ), in the + or negative z direction (or x or y depending on step 1), so:

$$\underline{M} = +/- |\underline{r}_{\perp}| |\underline{F}|$$

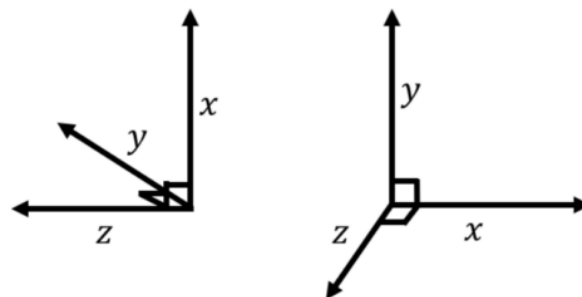
### 3.1.3 Three-Finger Configuration

If you find curling your fingers too confusing, you can try this method that uses your thumb, pointer finger, and middle finger all 90 degrees apart. Your thumb is x, your pointer finger is y, your middle finger is z.

This is done by using your right hand, aligning your thumb with the first vector and your index with the second vector. The cross product will point in the direction of your middle finger (when you hold your middle finger perpendicular to the other two fingers). This is illustrated in Figure A.14. Thus, you can often avoid using equation A.1 and instead use the right hand rule to determine the direction of the cross product and equation A.2 to find its



magnitude.



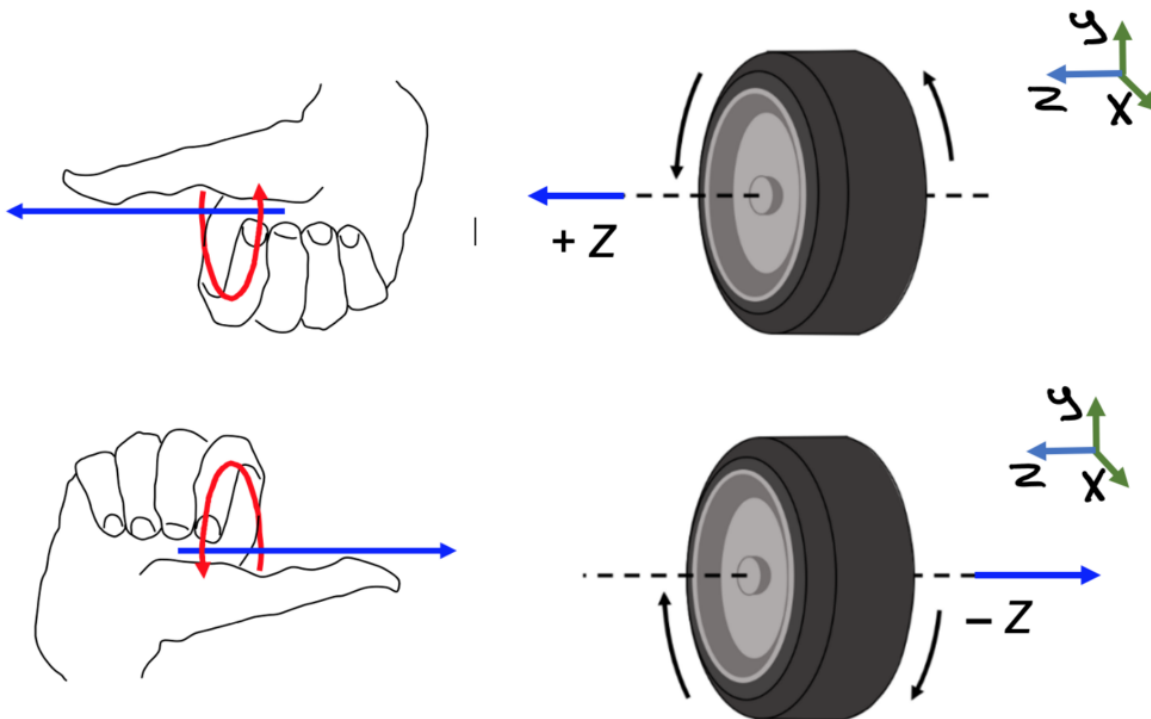
*Figure A.15: Two possible orientations for a three dimensional coordinate system. You can confirm using the right hand rule that the z axis is the cross product  $\vec{x} \times \vec{y}$ .*

Source: Introductory Physics, Ryan Martin et al., <https://openlibrary.ecampusontario.ca/catalogue/item/?id=4c3c2c75-0029-4c9e-967f-41f178bebbbbb>, page 823-825

## The “Curly Method”

For axial vectors, you use what I’m calling the curly method. To find whether the axis of rotation is positive or negative, curl your fingers in the direction of rotation and your thumb shows the direction of rotation, i.e. whether rotation is along the positive or negative x y or z direction. (This assumes you already have a coordinate frame defined to see which axis the wheel is rotating around and which direction).

If a wheel is rolling, the axis is what it rolls around. Curl your fingers in the direction of rotation and your thumb shows the direction of rotation.



### Key Takeaways

*Basically:* The right hand rule helps us to be consistent with how the  $x - y - z$  axes are oriented. It follows the rule that  $\mathbf{X} \times \mathbf{Y} = \mathbf{Z}$ . Using your fingers and thumb, there are two different methods. For one: point your fingers in the direction of  $x$ , curl them towards  $y$  (you may have to flip your hand), and your thumb shows the direction of  $z$ . Trying to copy this 3d image onto your 2d page may be difficult, but with practice you'll see the right angles between the drawn axes.

*Application:* How do I know which way to push on the torque wrench to make the bolt on my wheel turn? If I point my thumb in the direction I want the bolt to move, and curl my fingers around the direction of the threads, I can see whether to push or pull on the wrench.

*Looking Ahead:* We will calculate the moment many times throughout the rest of the book, and we need the right-hand rule every time especially as we get into Chapter 4 and Rigid Body Equilibrium Equations.

Hand from page 127 of Calculus Based Physics, Jeffrey W. Schnick,  
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=ce74a181-ccde-491c-848d-05489ed182e7> &  
tire from page 828 of Introductory Physics, Ryan Martin et al.,  
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=4c3c2c75-0029-4c9e-967f-41f178bebbbbb>,  
Edited by author.

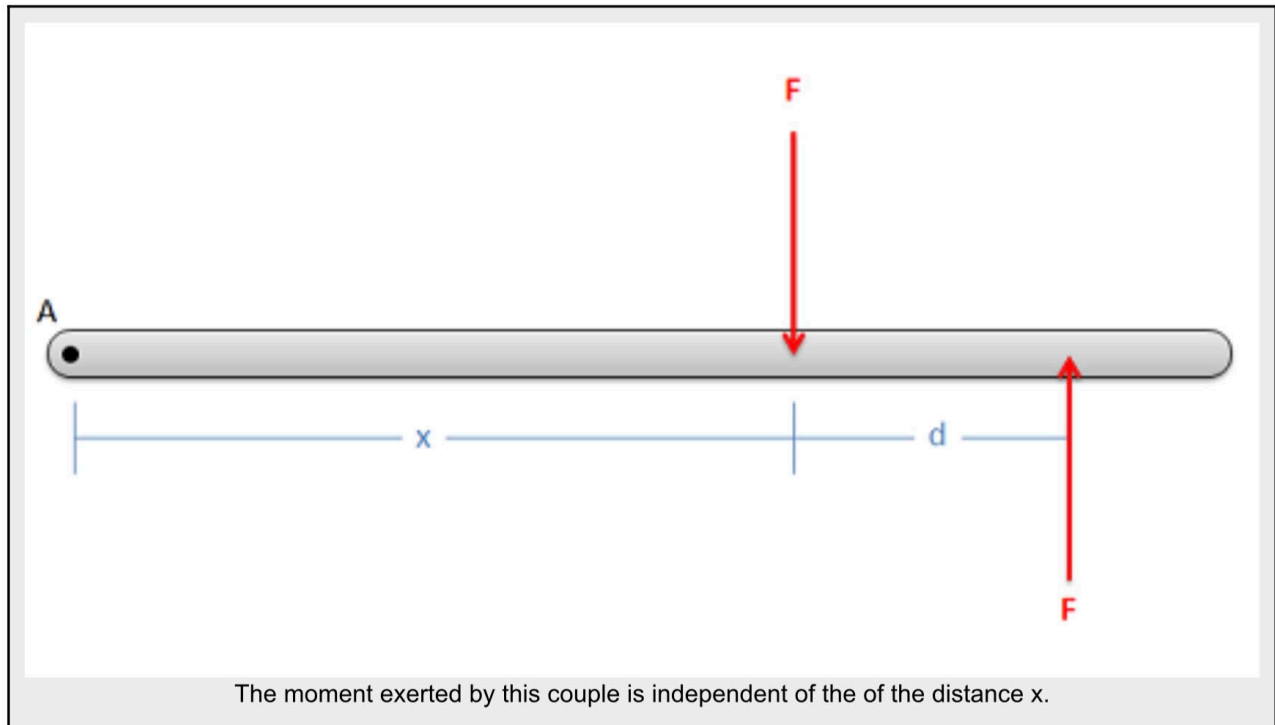
## 3.2 Couples

A **couple** is a set of equal and opposite forces that exerts a net moment on an object but no net force. Because the couple exerts a net moment without exerting a net force, couples are also sometimes called **pure moments**.



The two equal and opposite forces exerted on this lug wrench are a couple. They exert a moment on the lug nut on this wheel without exerting any net force on the wheel. Adapted from image by Steffen Heinz Caronna CC-BY-SA 3.0.

The moment exerted by a couple also differs from the moment exerted by a single force in that it is independent of the location you are taking the moment about. In the example below we have a couple acting on a beam. Each force has a magnitude  $F$  and the distance between the two forces is  $d$ .



Now we have some point A, which is distance  $x$  from the first of the two forces. If we take the moment of each force about point A, and then add these moments together for the net moment about point A we are left with the following formula.

$$M = -(F \cdot x) + (F \cdot (x + d))$$

If we rearrange and simplify the formula above, we can see that the variable  $x$  actually disappears from the equation, leaving the net moment equal to the magnitude of the forces ( $F$ ) times the distance between the two forces ( $d$ ).

$$M = -(F \cdot x) + (F \cdot x) + (F \cdot d) \implies M = (F \cdot d)$$

This means that no matter what value of  $x$  we have, the magnitude of the moment exerted by the couple will be the same. The magnitude of the moment due to the couple is independent of the location we are taking the moment about. This will also work in two or three dimensions as well. The magnitude of the moment due to a couple will always be equal to the magnitude of the forces times the perpendicular distance between the two forces.

Source: Engineering Mechanics, Jacob Moore et al.,  
[http://mechanicsmap.psu.edu/websites/3\\_equilibrium\\_rigid\\_body/3-3\\_couples/couples.html](http://mechanicsmap.psu.edu/websites/3_equilibrium_rigid_body/3-3_couples/couples.html)

### Key Takeaways

*Basically:* Couples are made from two forces in opposite directions that create a moment around an axis

*Application:* Turning the steering wheel of your car, you push one hand up and the other down to turn the wheel. To calculate the size of the couple, you multiply the force exerted by the distance between your hands (the diameter of the wheel).



*Looking Ahead:* While moments are more common in Ch 4 rigid body equations, it's important to know what couples are and how to find them.

## 3.3 Distributed Loads

### 3.3.1 Intensity

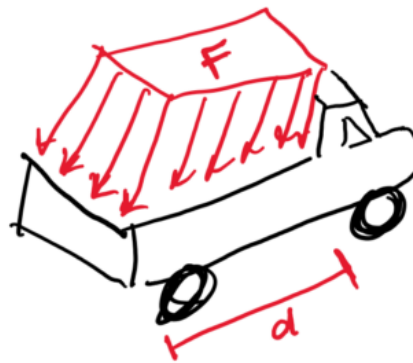
Distributed loads are a way to represent a force over a certain distance. Sometimes called *intensity*, given the variable:

$$\textbf{Intensity} \quad w = F / d \quad [=] \quad \text{N/m, lb/ft}$$

While pressure is force over area (for 3d problems), intensity is force over distance (for 2d problems). It's like a bunch of mattresses on the back of a truck. You can model it as 1 force acting at the center (an equivalent point load as in 3.3.2, or you can model it as intensity and divide the total force by the width of the truck bed (the distance that's not visible in this image).



Modelling this  
in 3d:

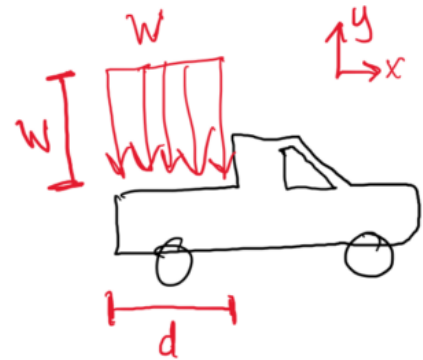


$$w = \frac{F}{d}$$

One 'slice' of  
the force is  
the intensity



Modelling this  
in 2d:



In 2d, one  
'slice' looks  
like:



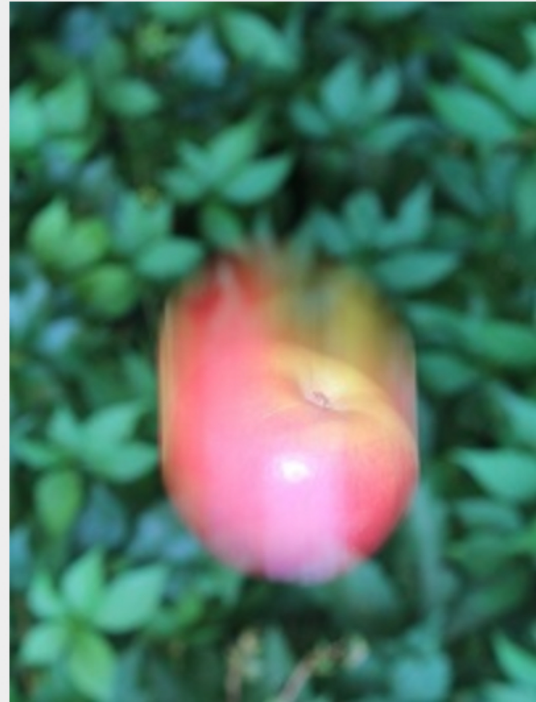
A distributed load is any force where the **point of application** of the force is an **area** or a **volume**. This means that the "point of application" is not really a point at all. Though distributed loads are more difficult to analyze than point forces, distributed loads are quite common in real world systems so it is important to understand how to

model them.

Distributed loads can be broken down into **surface forces** and **body forces**. Surface forces are distributed forces where the point of application is an area (a surface on the body). Body forces are forces where the point of application is a volume (the force is exerted on all molecules throughout the body). Below are some examples of surface and body forces.

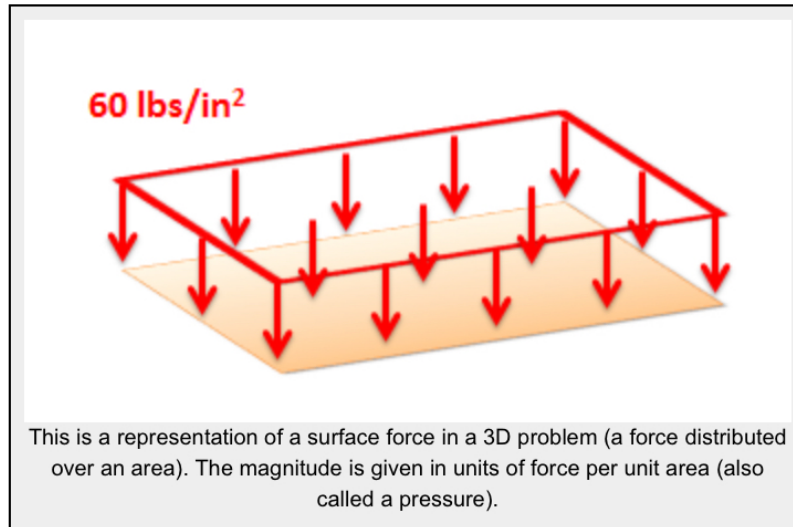
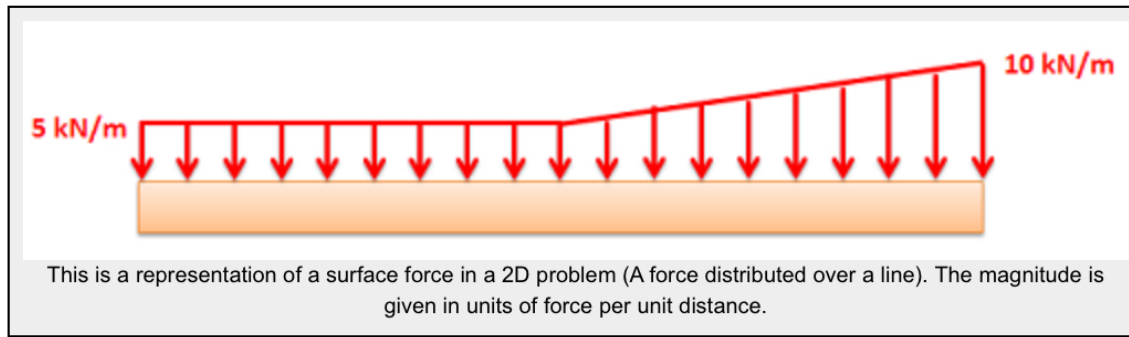


The water pressure pushing on the surface of this dam is an example of a surface force.  
Image by Curimedia CC-BY-SA 2.0



The gravitational force on this apple is distributed over the entire volume of the fruit. Gravitational forces are an example of body forces. Image by Zátonyi Sándor CC-BY 3.0

Distributed loads are represented as a field of vectors. This is drawn as a number of discrete vectors along a line, over a surface, or over a volume, that are connected with a line or a surface as shown below.



Though these representations show a discrete number of individual vectors, there is actually a magnitude and direction at all points along the line, surface, or body. The individual vectors represent a sampling of these magnitudes and directions.

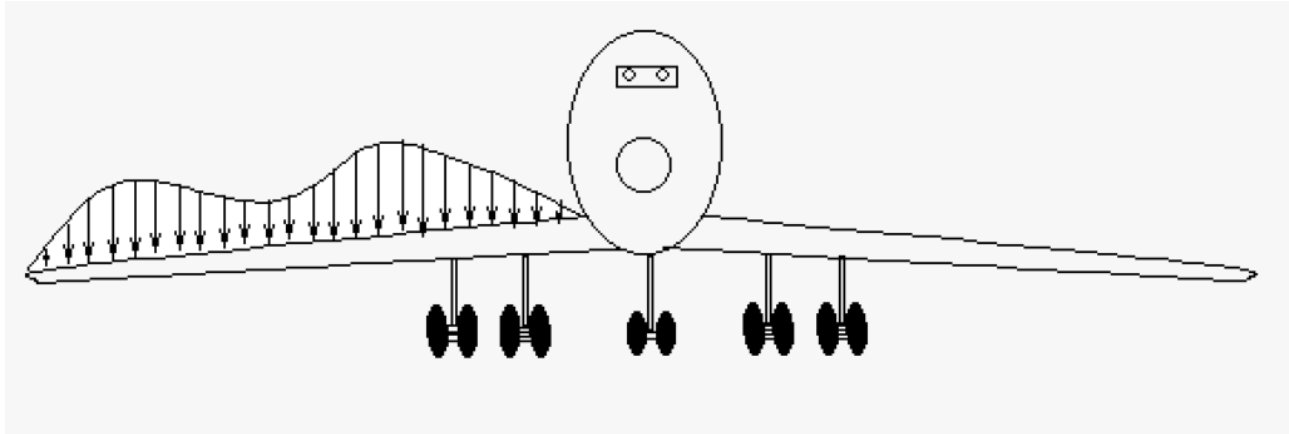
It is also important to realize that the magnitudes of distributed forces are given in force per unit distance, area, or volume. We must integrate the distributed load over its entire range to convert the force into the usual units of force.

## Analyzing Distributed Load:

For analysis purposes in statics and dynamics, we will usually substitute in a single point force that is statically equivalent to the distributed load in the problem. This single point force is called the **equivalent point load** and it will cause the same accelerations or reaction forces as the distributed load while simplifying the math.

Source: Engineering Mechanics, Jacob Moore et al.,  
[http://mechanicsmap.psu.edu/websites/4\\_statically\\_equivalent\\_systems/4-4\\_distributed\\_forces/distributedforces.htm](http://mechanicsmap.psu.edu/websites/4_statically_equivalent_systems/4-4_distributed_forces/distributedforces.htm)  
 1

An additional example:



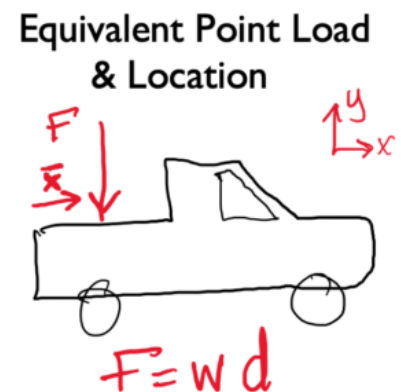
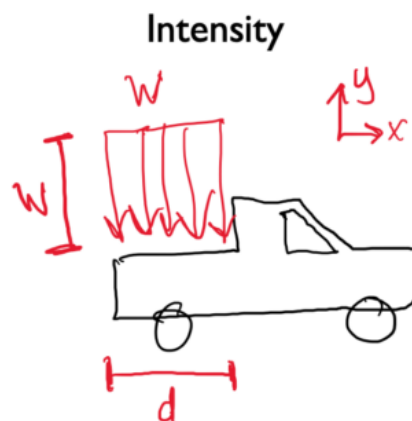
This is a more complex example of a distributed load. This is a cartoon of an airplane with its wing covered in a combination of snow and ice. In a real world situation loads will not accommodate people for ease of calculation, you get what you get. In this case we could approximate this shape with two semi-circles on each end of the wing with a triangle ( $\nabla$ ) in the middle. For more accuracy we could use a system similar to the trapezoidal rule.

Source: " Statics" by LibreTexts is licensed under CC BY-NC-SA .

[https://eng.libretexts.org/Bookshelves/Introduction\\_to\\_Engineering/EGR\\_1010%3A\\_Introduction\\_to\\_Engineering\\_for\\_Engineers\\_and\\_Scientists/14%3A\\_Fundamentals\\_of\\_Engineering/14.11%3A\\_Mechanics/14.11.01%3A\\_Statics](https://eng.libretexts.org/Bookshelves/Introduction_to_Engineering/EGR_1010%3A_Introduction_to_Engineering_for_Engineers_and_Scientists/14%3A_Fundamentals_of_Engineering/14.11%3A_Mechanics/14.11.01%3A_Statics)

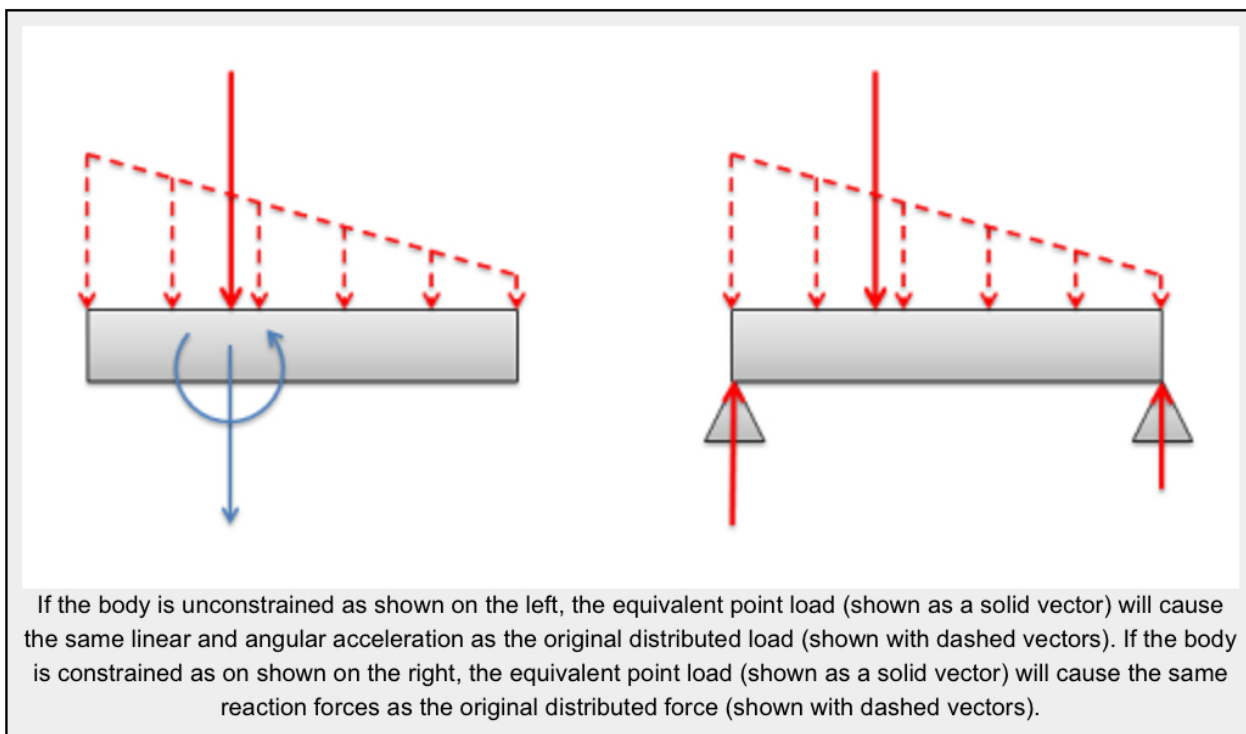
### 3.3.2 Equivalent Point Load & Location

Distributed loads can be modeled as a single point force that is located at the centroid of the object. You can use straight-forward algebra, or use integration for more complex shapes. Then you replace the distributed load with the single point load acting at  $x$  distance. See in the truck example:



There are two ways to calculate this, using integrals and using the area and centroid.

An equivalent point load is a **single point force** that will have the **same effect** on a body as the original loading condition, which is usually a distributed load. The equivalent point load should always cause the same linear acceleration and angular acceleration as the original load it is equivalent to (or cause the same reaction forces if the body is constrained). Finding the equivalent point load for a distributed load often helps simplify the analysis of a system by removing the integrals from the equations of equilibrium or equations of motion in later analysis.



## Finding the Equivalent Point Load

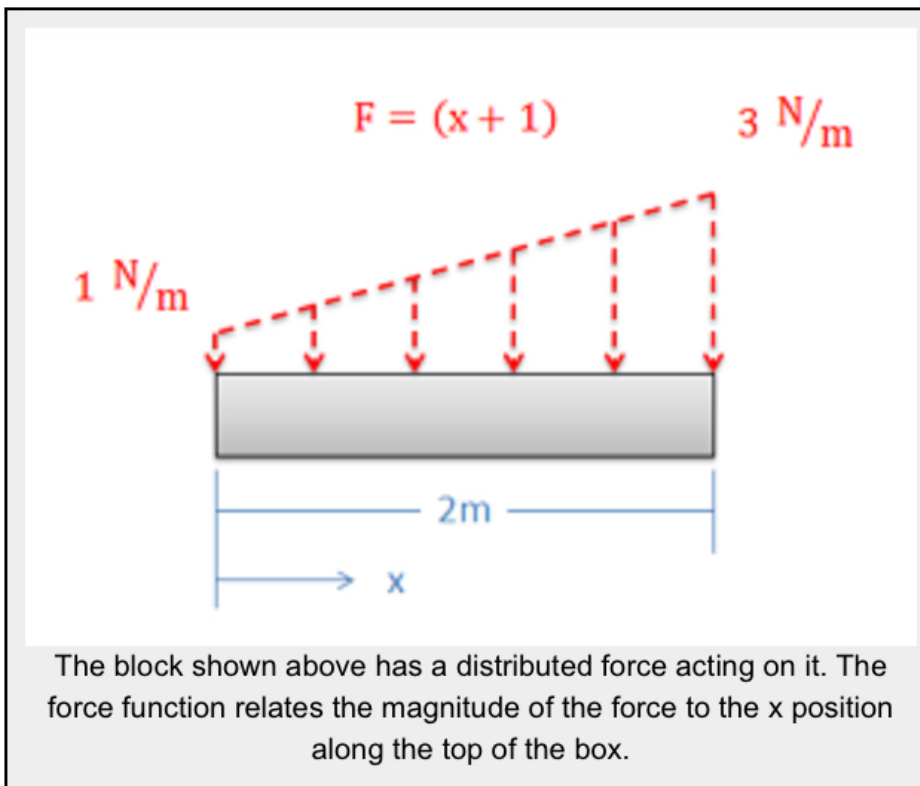
When finding the equivalent point load we need to find the magnitude, direction, and point of application of a single force that is equivalent to the distributed load we are given. In this course we will only deal with distributed loads with a uniform direction, in which case the direction of the equivalent point load will match the uniform direction of the distributed load. This leaves the magnitude and the point of application to be found. There are two options available to find these values:

1. We can find the magnitude and the point of application of the equivalent point load **via integration** of the force functions.
2. We can use the **area/volume and the centroid/center of volume** of the area or volume under the force function.

The first method is more flexible, allowing us to find the equivalent point load for any force function that we can make a mathematical formula for (assuming we have the skill in calculus to integrate that function). The second method is usually faster, assuming that we can look up the values for the area or volume under the force curve and the values for the centroid or center of volume for the area under the curve.

## Using Integration in 2D Surface Force Problems:

Finding the equivalent point load via integration always begins by determining the mathematical formula that is the **force function**. The force function mathematically relates the magnitude of the force ( $F$ ) to the position ( $x$ ). In this case the force is acting along a single line, so the position can be entirely determined by knowing the  $x$  coordinate, but in later problems we may also need to relate the magnitude of the force to the  $y$  and  $z$  coordinates. In our example to the left, we can relate magnitude of the force to the position by stating that the magnitude of the force at any point in Newtons per meter is equal to the  $x$  position in meters plus one.



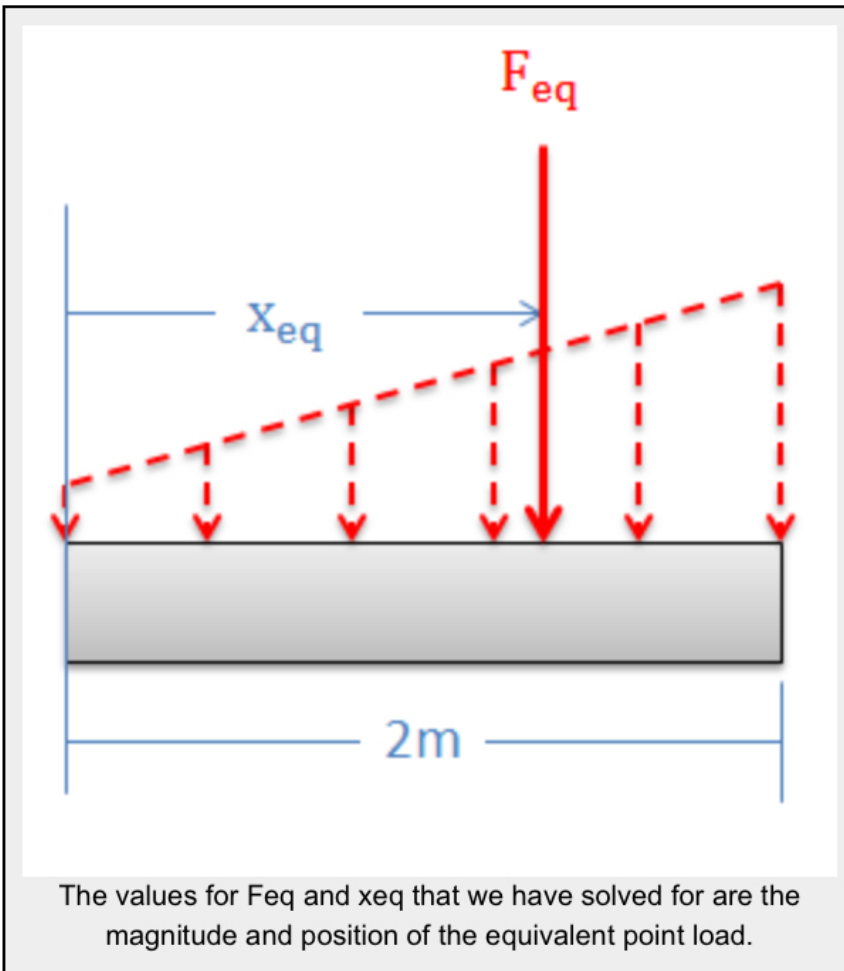
The magnitude of the equivalent point load will be equal to the area under the force function. This will be the integral of the force function over its entire length (in this case from  $x = 0$  to  $x = 2$ ).

$$F_{eq} = \int_{x_{min}}^{x_{max}} F(x) dx$$

Now that we have the magnitude of the equivalent point load such that it matches the magnitude of the original force, we need to adjust the position ( $x_{eq}$ ) such that it would cause the same **moment** as the original distributed force. The moment of the distributed force will be the integral of the force function ( $F(x)$ ) times the moment arm about the origin ( $x$ ). The moment of the equivalent point load will be equal to the magnitude of the equivalent point load that we just found times the moment arm for the equivalent point load ( $x_{eq}$ ). If we set these two things equal to one another and then solve for the position of the equivalent point load ( $x_{eq}$ ) we are left with the following equation:

$$x_{eq} = \frac{\int_{x_{min}}^{x_{max}} (F(x) \cdot x) dx}{F_{eq}}$$

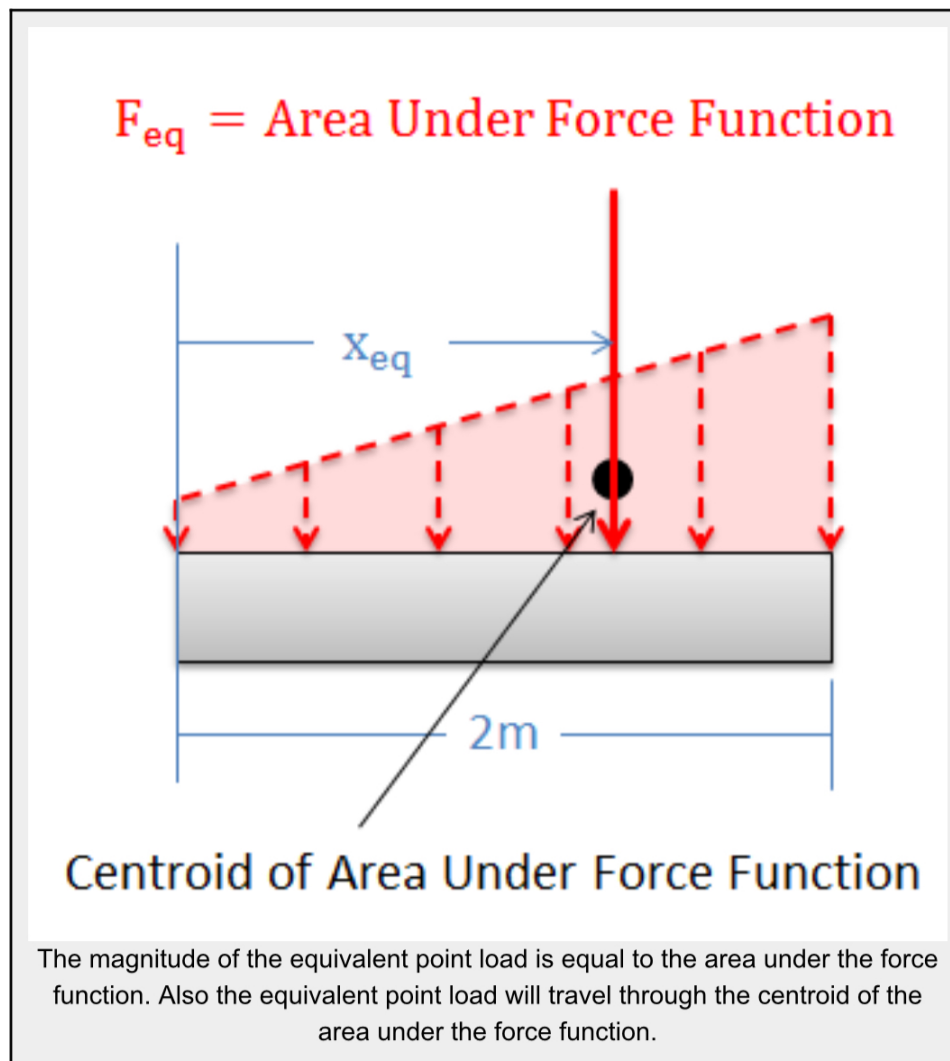
Now that we have the magnitude, direction, and position of the equivalent point load, we can draw the point load in our original diagram. This point force can be used in place of the distributed force in further analysis.



## Using the Area and Centroid in 2D Surface Force Problems:

As an alternative to using integration, we can use the area under the force curve and the centroid of the area under the force curve to find the equivalent point load's magnitude and point of application respectively.



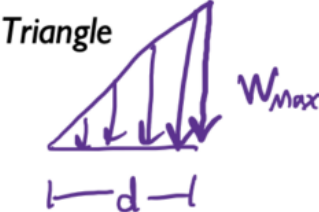


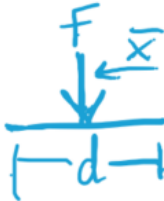
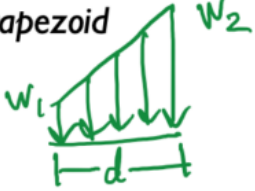
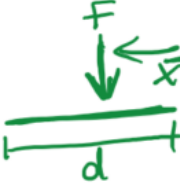


The **magnitude**( $F_{eq}$ ) of the equivalent point load will be equal to the **area under the force function**. We can find this area using calculus, but there are often easier geometry based ways of finding the area under the force function.

The equivalent point load will also **travel through centroid of the area under the force function**. This allows us to find the value for  $x_{eq}$ . The centroid for many common shapes can be looked up in tables, and the parallel axis theorem can be used to determine the centroid of more complex shapes (see the centroid page for more details).

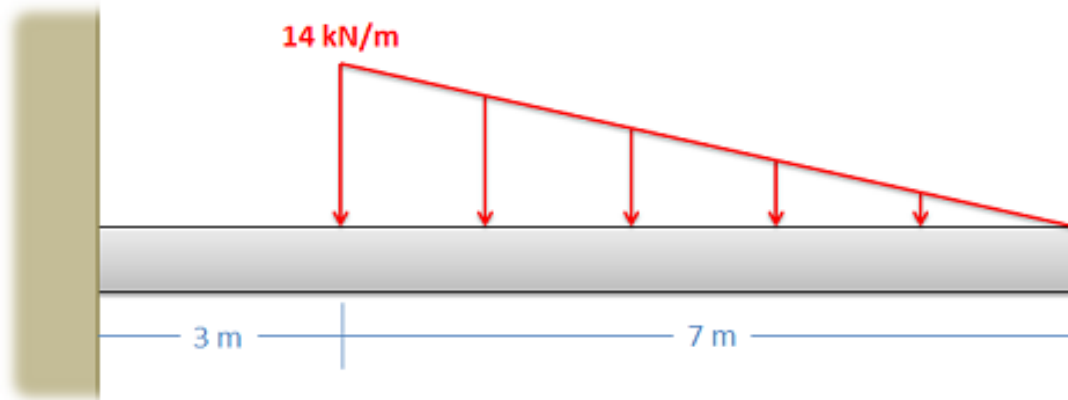
Source: Engineering Mechanics, Jacob Moore et al.,  
[http://mechanicsmap.psu.edu/websites/4\\_statically\\_equivalent\\_systems/4-5\\_equivalent\\_point\\_load/equivalentpointload.html](http://mechanicsmap.psu.edu/websites/4_statically_equivalent_systems/4-5_equivalent_point_load/equivalentpointload.html)

Here are the equations for some common shapes:

Shape	Equivalent Point Load	Force Equation	Location
<b>Triangle</b> 		$F = \frac{1}{2}bh = \frac{1}{2}dw_{max}$	$\bar{x} = d/3$
<b>Square</b> 		$F = bh = wd$	$\bar{x} = d/2$
<b>Trapezoid</b> 		$F = \frac{1}{2}b(h_1 + h_2)$ $= \frac{1}{2}d(w_1 + w_2)$	$\bar{x} = \frac{d(w_1 + 2w_2)}{3(w_1 + w_2)}$

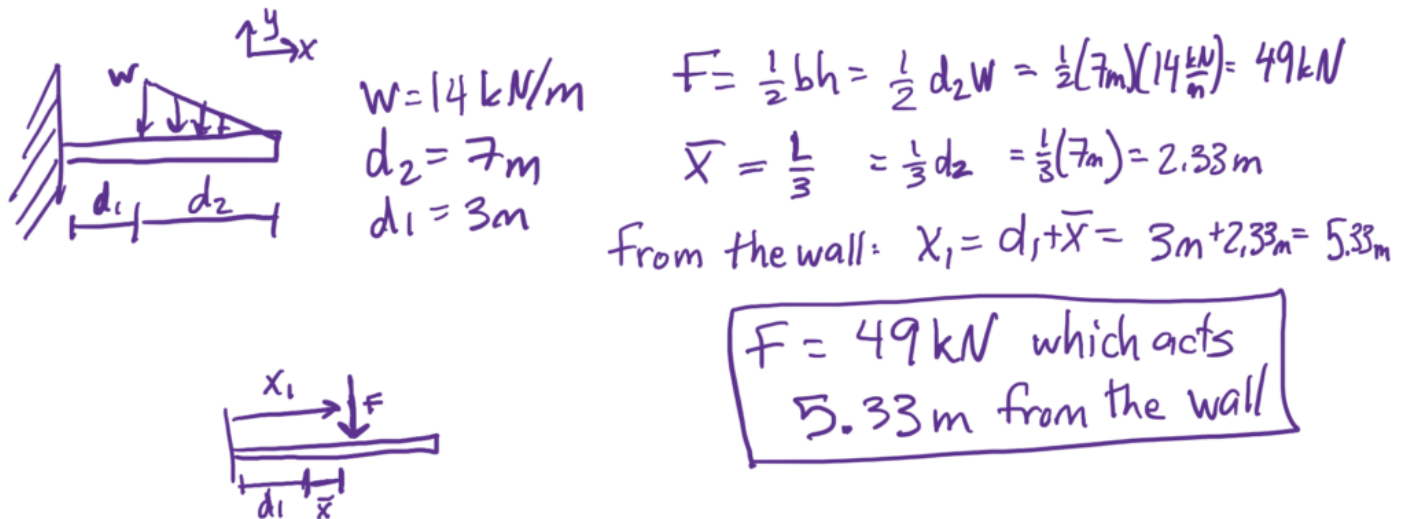
Example 1: Equivalent force and location:

What is the resultant force and where does it act from the wall?



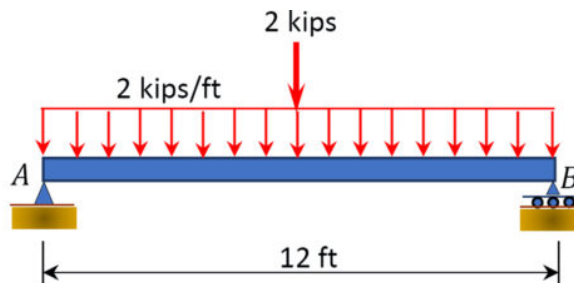
Source:

[http://mechanicsmap.psu.edu/websites/4\\_statically\\_equivalent\\_systems/4-5\\_equivalent\\_point\\_load](http://mechanicsmap.psu.edu/websites/4_statically_equivalent_systems/4-5_equivalent_point_load)

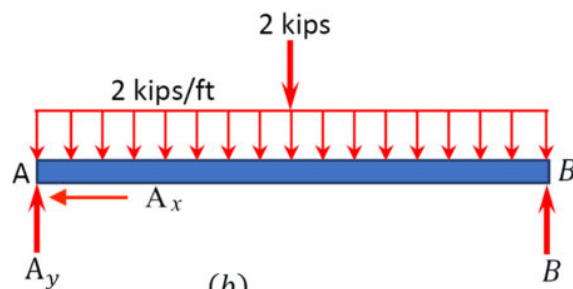


See solution here using integration from Engineering Mechanics, Jacob Moore et al., [http://mechanicsmap.psu.edu/websites/4\\_statically\\_equivalent\\_systems/4-5\\_equivalent\\_point\\_load/pdf/EquivalentPointLoad\\_WorkedExample1.pdf](http://mechanicsmap.psu.edu/websites/4_statically_equivalent_systems/4-5_equivalent_point_load/pdf/EquivalentPointLoad_WorkedExample1.pdf)

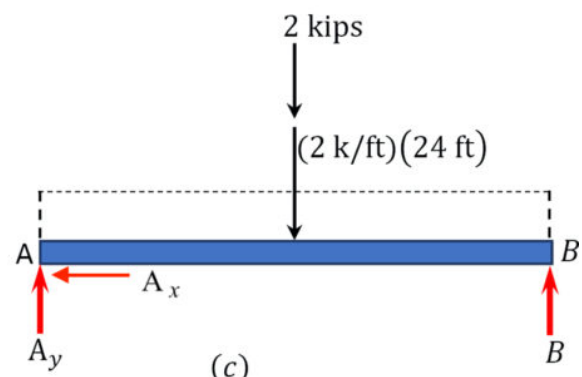
Example 2 (note: 1 kip = 1000 lb):



(a)

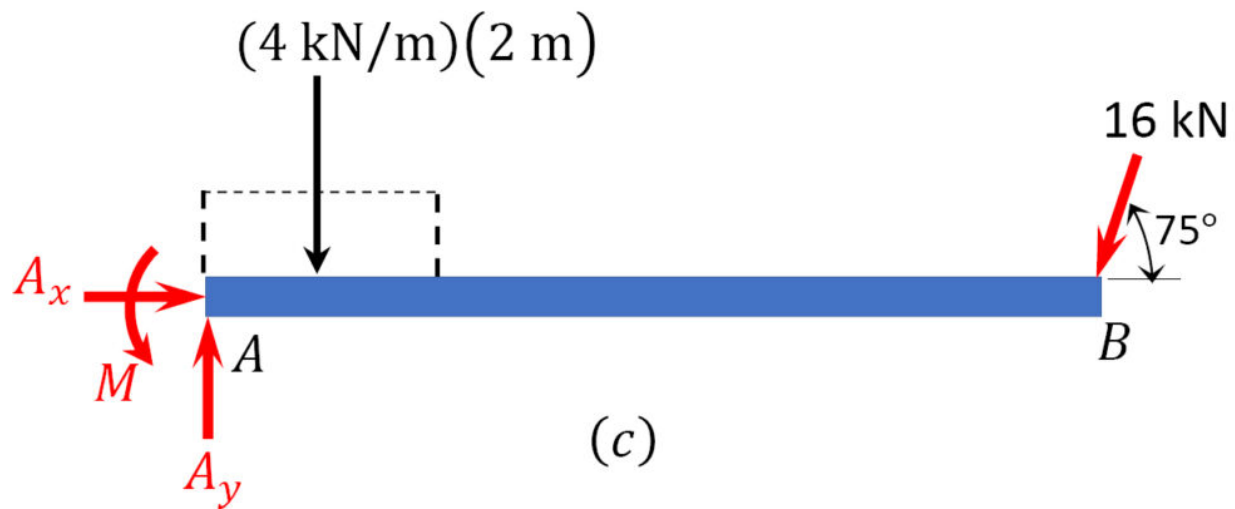
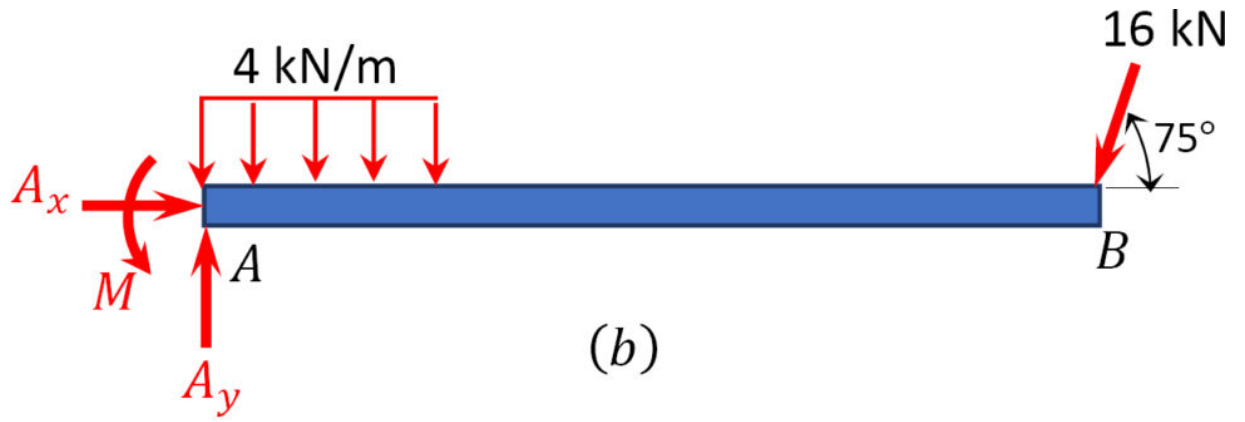


(b)

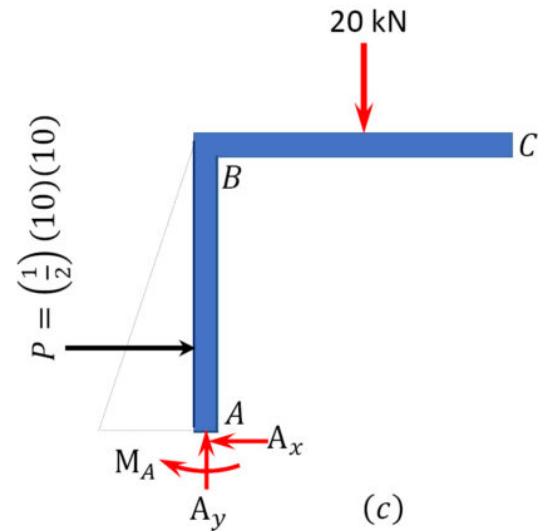
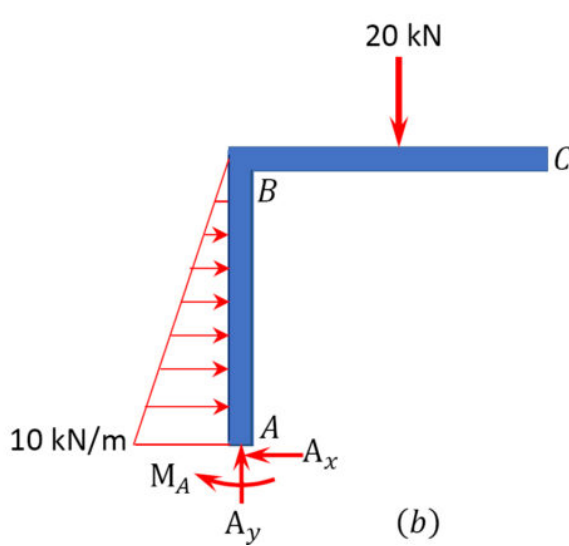
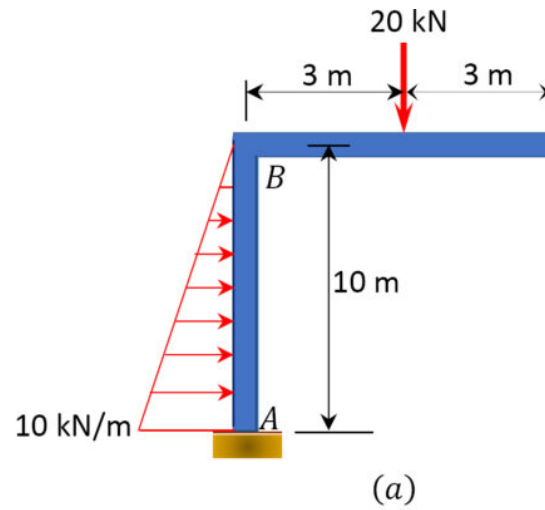


(c)

Example 3:



Example 4:



Source: "Equilibrium Structures, Support Reactions, Determinacy and Stability of Beams and Frames" by LibreTexts is licensed under CC BY-NC-ND .

[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.03%3A\\_Equilibrium\\_Structures\\_Support\\_Reactions\\_Determinacy\\_and\\_Stability\\_of\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.03%3A_Equilibrium_Structures_Support_Reactions_Determinacy_and_Stability_of_Beams_and_Frames)



Diagram illustrating the decomposition of a distributed load into two parts: a triangular load of width  $d_1$  and a rectangular load of width  $d_2$ , both with intensity  $w$ .

The total force is given by:

$$F = F_1 + F_2 = \frac{1}{2}b_1h_1 + b_2h_2 = \frac{1}{2}d_1w + d_2w$$

The center of mass is determined by the weighted average of the centers of the two component loads:

$$\bar{x} = \frac{\sum F_x}{\sum F} = \frac{F_1x_1 + F_2x_2}{F_1 + F_2} = \frac{\left(\frac{1}{2}d_1w\right)\left(d_2 + \frac{1}{2}d_1\right) + (d_2w)\left(\frac{1}{3}d_2\right)}{\left(\frac{1}{2}d_1 + d_2\right)w}$$

### Key Takeaways

Basically: Distributed loads are a way to model forces in 2d.  $\mathbf{F} = \mathbf{w} \mathbf{d}$  Sometimes called intensity, distributed loads have units of force over distance: N/m or lb/ft.

Application: For a truck carrying a heavy uneven load, find where the center of the force is.

Looking ahead: Distributed load helps to model uneven loads. We'll see it again as we do beam analysis

Image of truck from:

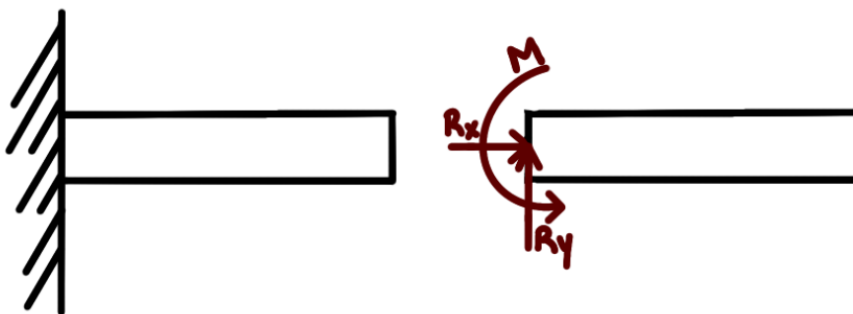
<https://get.pxhere.com/photo/car-transport-truck-vehicle-market-mattress-full-load-small-business-rwanda-overload-pickup-truck-overfull-automobile-make-612534.jpg>

## 3.4 Reactions & Supports

Imagine a beam extending from the wall. How much weight can the beam handle before it breaks away or falls 'off' the wall? It depends on the way it's attached to the wall. We model these real world situations using forces and moments. For example, the grand canyon skywalk lets people walk out over the grand canyon. You want to be sure that the skywalk is so the people on it are safe.



We call the skywalk a *cantilever beam* and turn the real world beam into a 2d model with constraints. So we can use the same terminology, it is a *fixed* constraint, preventing horizontal movement, vertical movement, and rotation.



Reaction forces and moments are how we model constraints on structures. They are external forces. There are 3 different kinds of constraints we will focus on in this course and they each have different



reaction forces and moments:

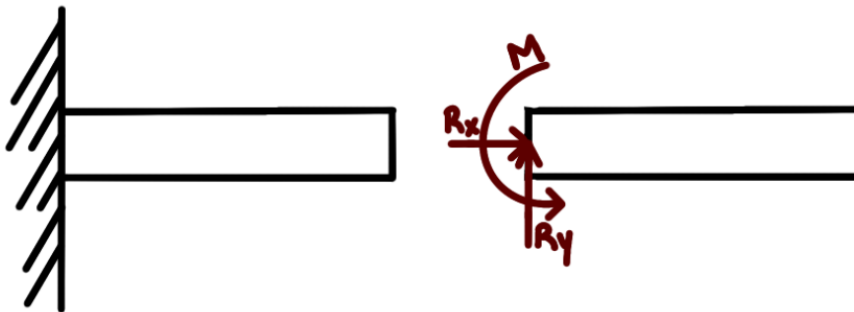


### 1. Pinned (Frictionless)

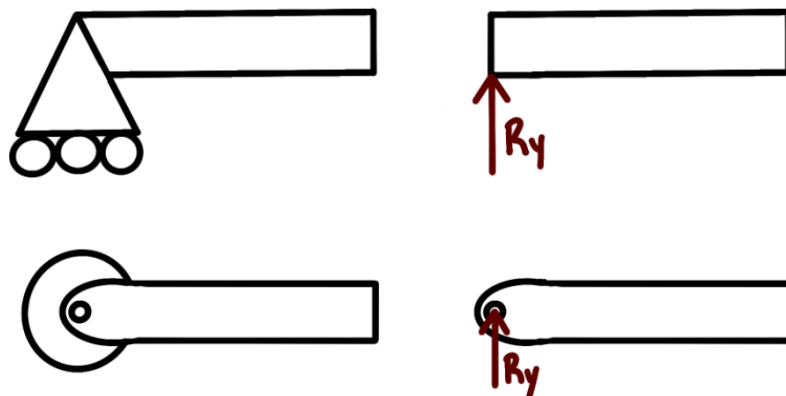
- Two reaction forces acting perpendicularly in the x and y directions.
- Pinned constraint and then its free body diagram shown:

### 2. Fixed

- Two reaction forces acting perpendicularly in the x and y directions



- Moment rotating about fixed constraint (usually a wall), use right hand rule to find its direction
- This is also called a cantilever beam.
- Fixed constraint and then FBD shown

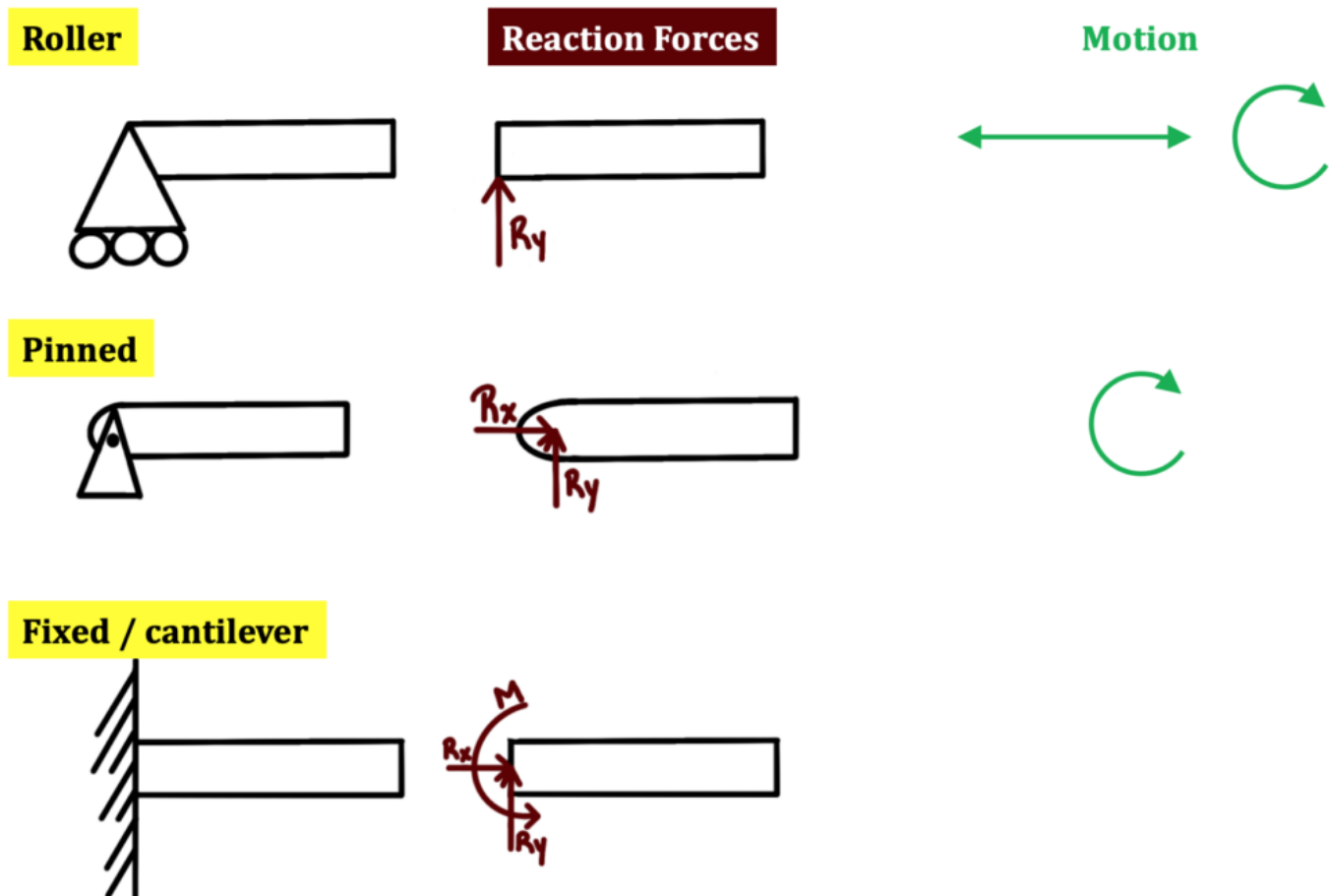


### 3. Roller (there are multiple kinds)

- Single reaction force acting in the y direction
- No moment is created
- This can be the ground that the object rests on as well
- Free body diagram shown for roller

Notice that the Fixed restraint is the most restrictive and the roller is the least restrictive. You put a force to show how the restraint restricts motion. The roller only keeps the object from moving vertically, so there is only 1 force. The pinned restraint doesn't allow horizontal or vertical movement, hence the two forces. The fixed beam restricts vertical translation, horizontal translation, and rotation, so there is a moment and two forces. Note that this applies only to 2d restraints.

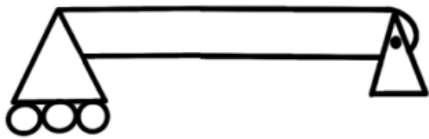
Here is a summary showing what motion is allowed by that type of constraint:



Typically reaction forces are either as follows: a pinned and a fixed reaction force together (1 reaction force + 2 reaction forces = 3 restraints) or a fixed beam (2 reaction forces and 1 moment = 3 restraints).

**Roller**

**Pinned**



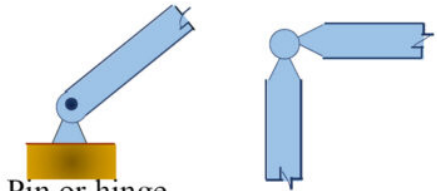
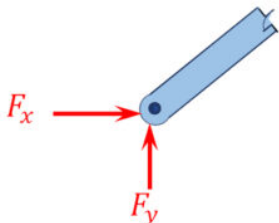
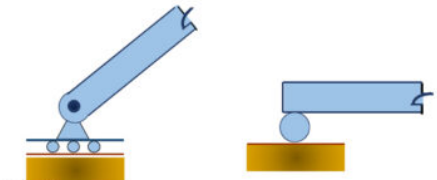
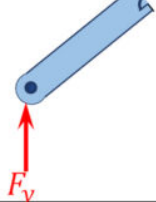
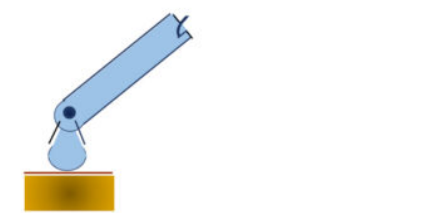
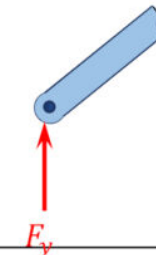
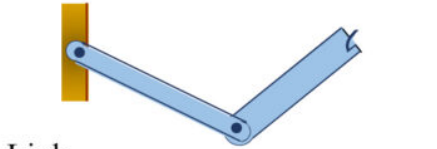
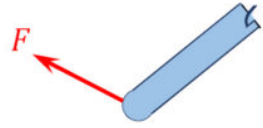
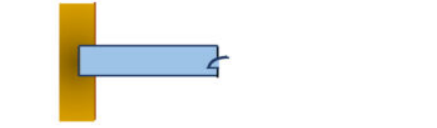
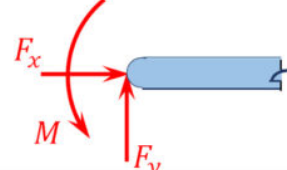
**Fixed / cantilever**



The information shown here is to model 2d situations. We don't get into 3d problems in this statics course, needless to say, there are more reaction forces and moments involved in 3-dimensions instead of 2 dimensions. The following section provides a second explanation on reactions & supports:

#### 3.4.1 Pin or Hinge Support

A pin support allows rotation about any axis but prevents movement in the horizontal and vertical directions. Its idealized representation and reactions are shown in Table 3.1:

Idealization of Support	Reaction	Characteristics
 Pin or hinge		Prevents movement in the vertical and horizontal direction but allows rotation.
 Roller		Prevents movement in the vertical direction but allows rotation and translation in the horizontal direction.
 Rocker		The characteristics of a rocker support are similar to that of a roller.
 Link		Prevents movement in the direction perpendicular to the axis of the link.
 Fixed		Does not allow translation in any direction and rotation.

### 3.4.2 Roller Support

A roller support allows rotation about any axis and translation (horizontal movement) in any direction parallel to the surface on which it rests. It restrains the structure from movement in a vertical direction. The idealized representation of a roller and its reaction are also shown in Table 3.1.

### 3.4.3 Rocker Support

The characteristics of a rocker support are like those of the roller support. Its idealized form is depicted in Table 3.1.

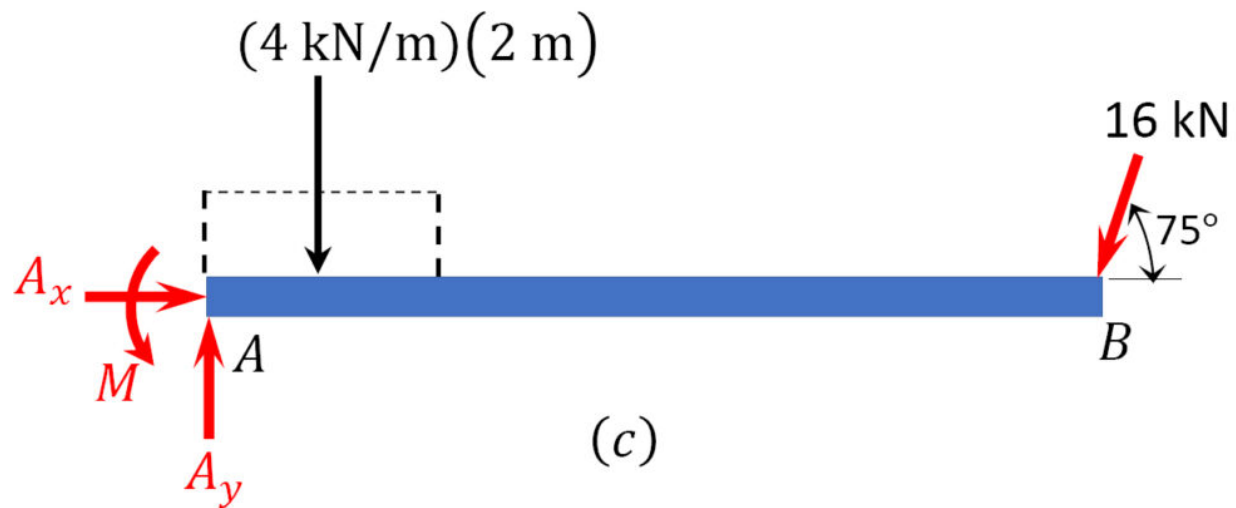
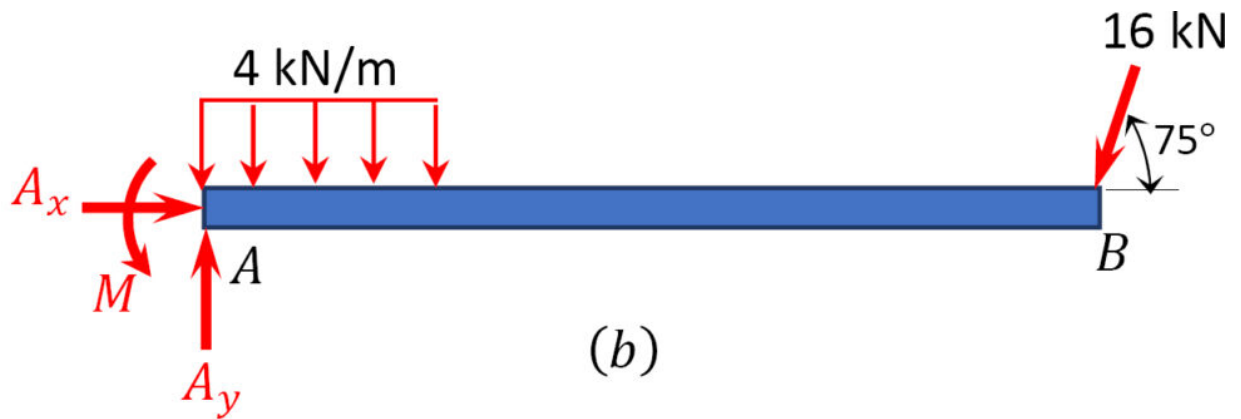
### 3.4.4 Link

A link has two hinges, one at each end. It permits movement in all direction, except in a direction parallel to its longitudinal axis, which passes through the two hinges. In other words, the reaction force of a link is in the direction of the link, along its longitudinal axis.

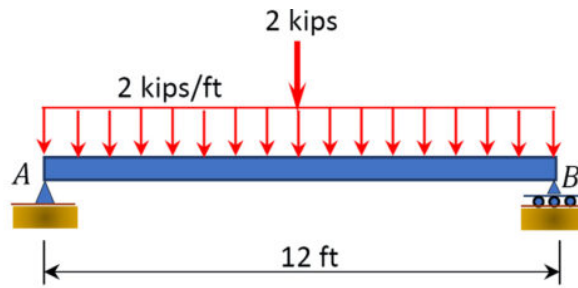
### 3.4.5 Fixed Support

A fixed support offers a constraint against rotation in any direction, and it prevents movement in both horizontal and vertical directions.

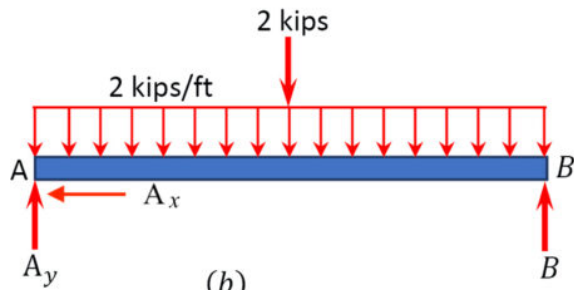
Example 1:



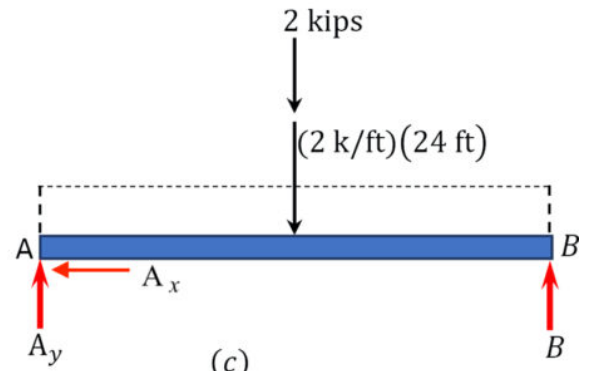
Example 2 ( $A_x$  added even though it turns out to be 0):



(a)

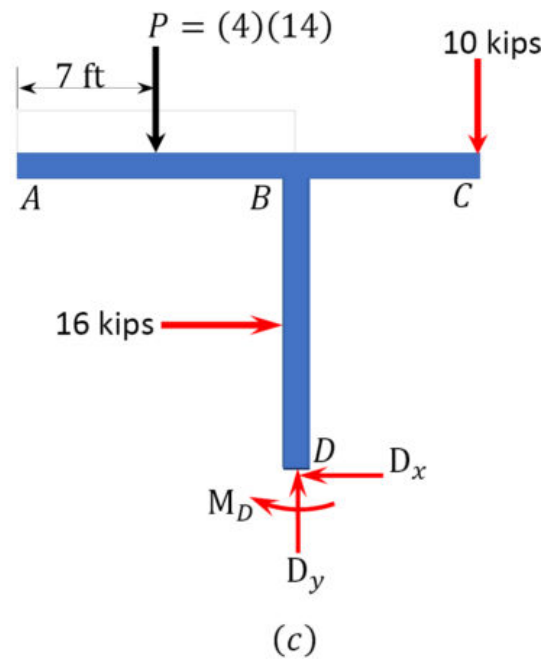
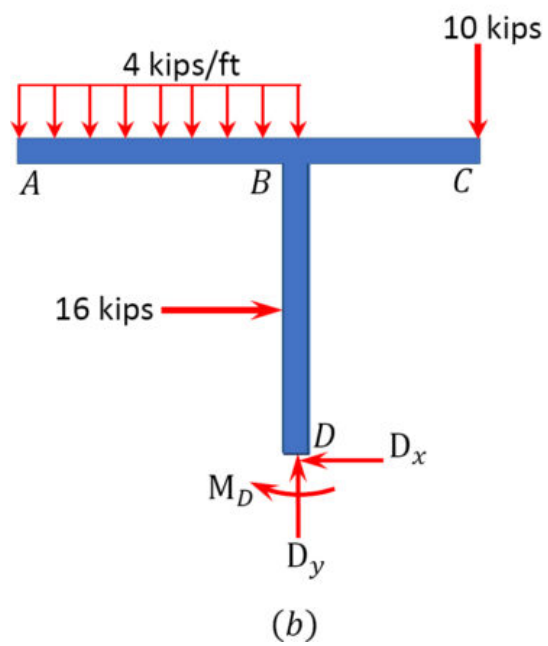
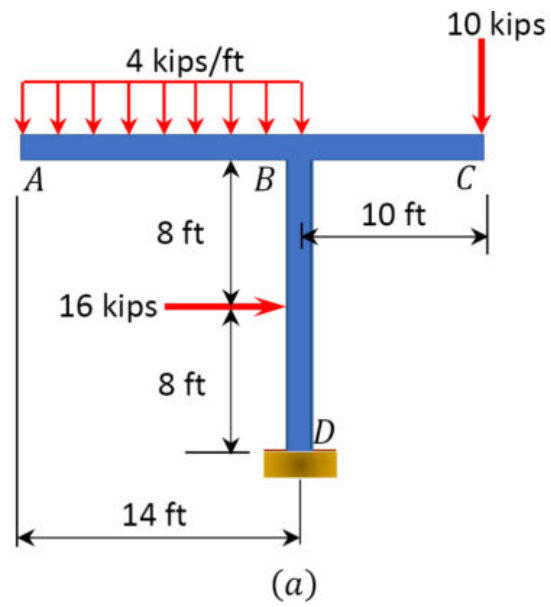


(b)

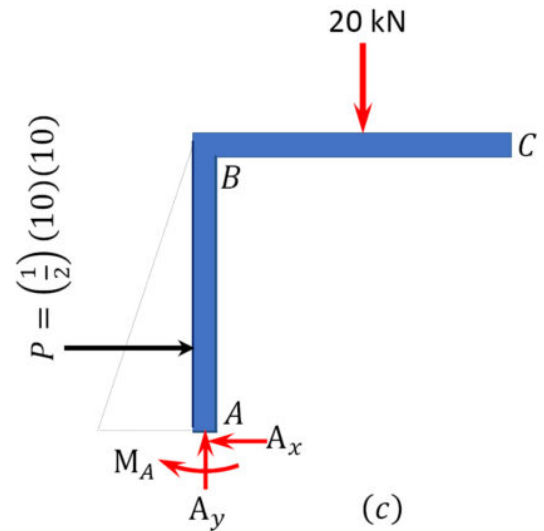
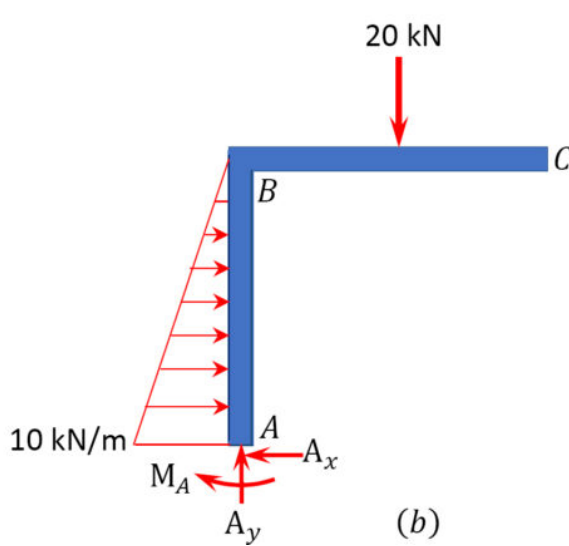
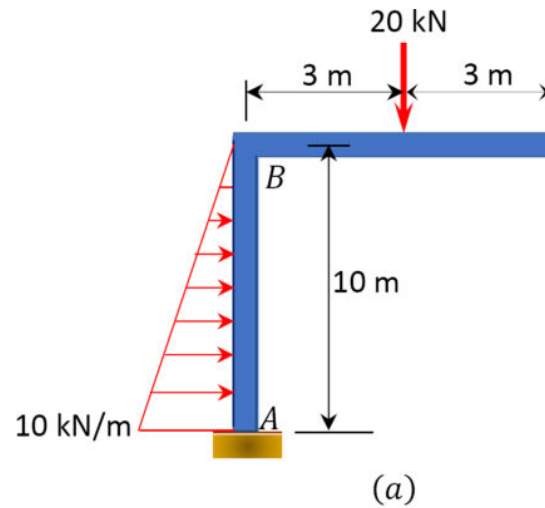


(c)

Example 3:



Example 4:



Source: "Equilibrium Structures, Support Reactions, Determinacy and Stability of Beams and Frames" by LibreTexts is licensed under CC BY-NC-ND .

[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.03%3A\\_Equilibrium\\_Structures\\_Support\\_Reactions\\_Determinacy\\_and\\_Stability\\_of\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.03%3A_Equilibrium_Structures_Support_Reactions_Determinacy_and_Stability_of_Beams_and_Frames)



*Basically:* Reaction forces and moments (or constraints) show how motion is restricted, here that is in 2 dimensions.

*Application:* A beam attached to the wall has three ways of restricting the motion: horizontal, vertical, and rotational.

*Looking Ahead:* Every time we model an scenario, we will use reaction forces to show what type of motion is being restrained. In Chapter 4, we will be able to calculate the reaction forces/moments.

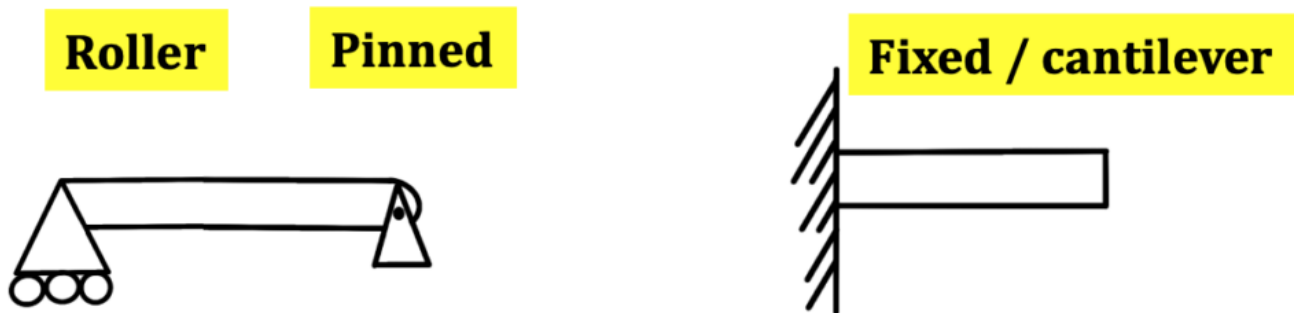
Written by Gayla & Libby

## 3.5 Indeterminate Loads

### *Determinate Loads*

Once you have your equilibrium equations, you can solve them for unknowns using algebra. The number of unknowns that you will be able to solve for will be the number of equilibrium equations that you have. In the x-y-z coordinate frame, there are 3 equations. so there can be 3 unknowns. These are statically determinate.

Typically reaction forces are either as follows: a pinned and a fixed reaction force together (1 reaction force + 2 reaction forces = 3 restraints) or a fixed beam (2 reaction forces and 1 moment = 3 restraints).



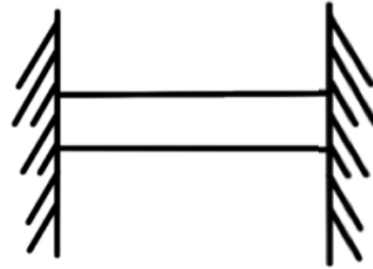
### *Indeterminate Loads*

When you have more unknowns than equations, the problem is a **statically indeterminate problem** and you will need additional information to solve for the given unknowns. You'll learn how to model and solve for these problems in your Structures course, but for Statics you need to be able to identify what is determinate and what is indeterminate.

Essentially, a problem is statically indeterminate if there are more unknown variables than there are equations you can use to solve for the unknowns. This means it cannot be solved using equilibrium equations alone. You would need to simplify the problem or make an assumption for it to be solved. You'll look into that more in Structures, because in real life, we want redundancy. We want to be sure that structures are strong and one part can fail but that the whole building doesn't collapse. However, we cannot over-constrain something so that it breaks because it has no way to expand during heat or

cooling. All of this makes it harder to model, but safer to use.

Here are examples of statically indeterminate problems:



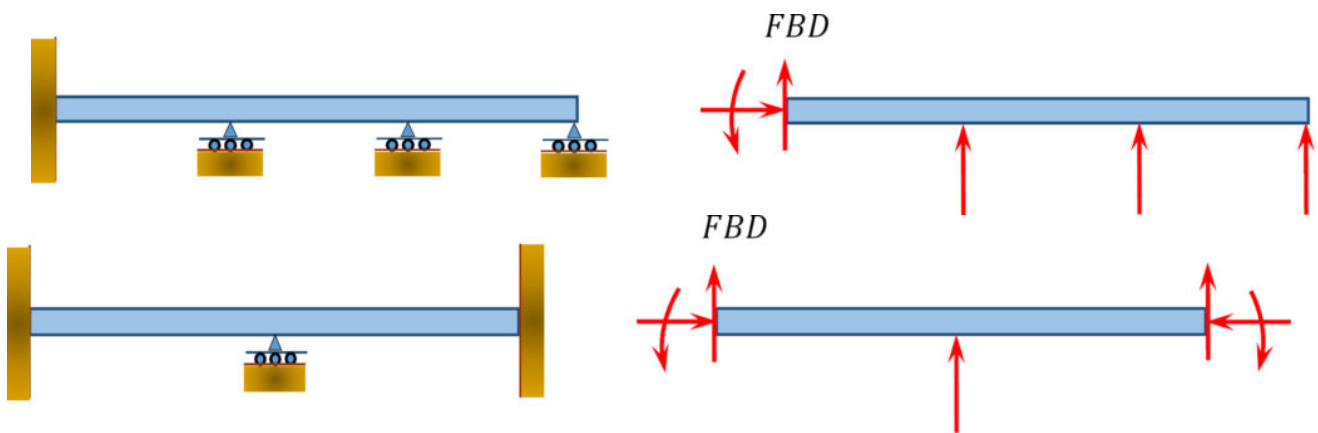
Here is a real world example of beams. Notice the breaks between sections (in the yellow circles) to allow for expansion and contraction. The top beam in between yellow circles would be an example of a pin and roller system where one side is pinned and the other side is a roller allowing for horizontal expansion.



Source:

<https://www.maxpixel.net/Guangdong-Structure-Shenzhen-Bridge-Metro-Station-5998185>

Examples of statically indeterminate structures:



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[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.03%3A\\_Equilibrium\\_Structures\\_Support\\_Reactions\\_Determinacy\\_and\\_Stability\\_of\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.03%3A_Equilibrium_Structures_Support_Reactions_Determinacy_and_Stability_of_Beams_and_Frames)

### Key Takeaways

*Basically:* When you have more unknowns than equations, the problem is a statically indeterminate problem

*Application:* Most situations are statically indeterminate, such as how beams are supported to provide for redundancy, thus we make assumptions to model a problem using equilibrium equations.

*Looking Ahead:* In Structures you will learn how to solve for statically indeterminate problems. In Statics, you need to be able to identify them.

Written by Gayla & Libby

## 3.6 Examples

Here are examples from Chapter 3 to help you understand these concepts better. These were taken from the real world and supplied by FSDE students in Summer 2021. If you'd like to submit your own examples, please send them to the author [eosgood@upei.ca](mailto:eosgood@upei.ca).

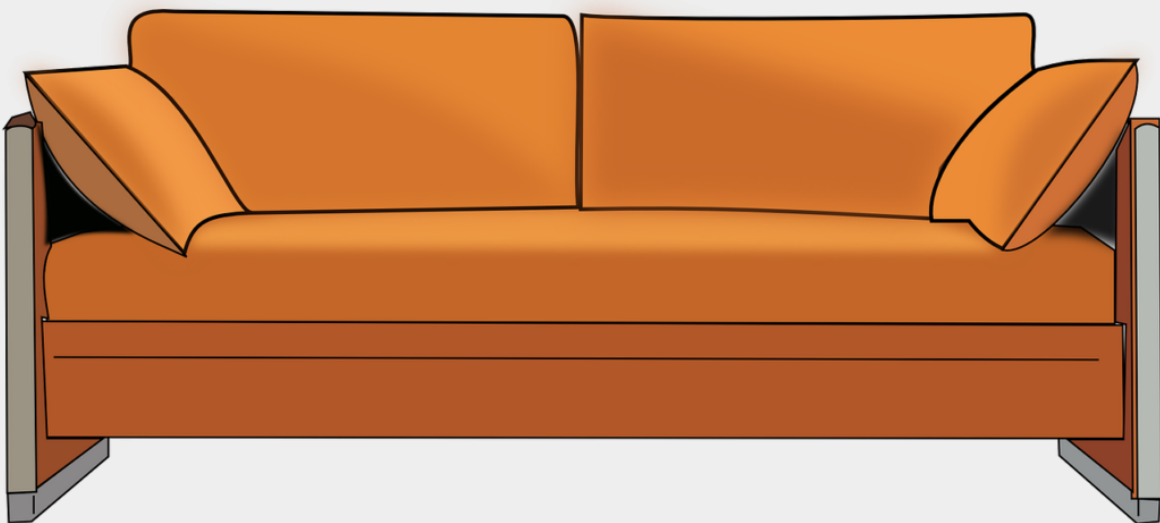
### Example 3.6.1: Reaction Forces, Submitted by Andrew Williamson

#### 1. Problem

A family is sitting watching TV on their couch. The couch is 5 m long and weighs 120 N. The child is sat 1 m away from one end and has a mass of 30 kg. The mother is sat 0.5 m away from the child and has a mass of 60 kg. The father is 3 m away from the mother and has a mass of 70 kg.

- Draw a free-body diagram of the couch
- Calculate the reaction force on each of the two legs.

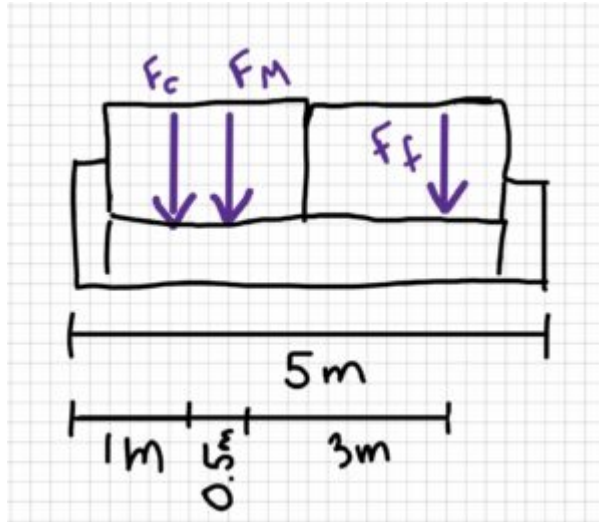
Assume the couch is supported by two rollers.



Source: <https://www.maxpixel.net/Seat-Couch-Interior-Home-Furniture-Room-Sofa-42817>

**2. Draw**

Sketch:

**3. Knowns and Unknowns**

Knowns:

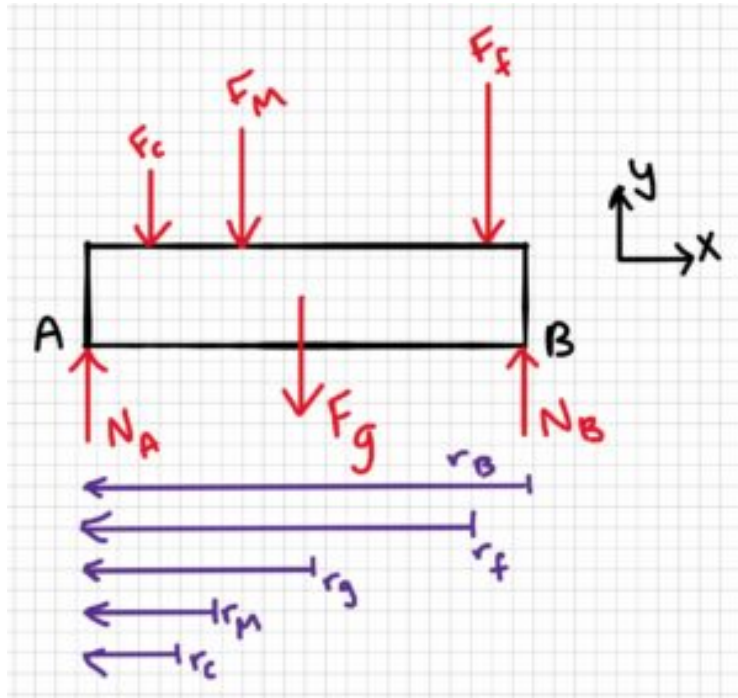
- $g = 9.81 \text{ m/s}^2$
- $m_c = 30 \text{ kg}$
- $m_M = 60 \text{ kg}$
- $m_f = 70 \text{ kg}$
- $F_g = 120 \text{ N}$
- $r_c = 1 \text{ m}$
- $r_M = 1.5 \text{ m}$
- $r_f = 4.5 \text{ m}$
- $r_B = 5 \text{ m}$
- $r_g = 2.5 \text{ m}$

Unknowns:  $N_A$ ,  $N_B$ **4. Approach**

Use equilibrium equations

**5. Analysis**

Part a:



Part b:

$$\begin{aligned} \sum M_B = 0 &= N_B \cdot r_B - F_c \cdot r_c - F_M \cdot r_M - F_f \cdot r_f - F_g \cdot r_g \\ N_B(5\text{m}) &= (30\text{kg} \cdot 9.81\text{m/s}^2)(1\text{m}) + (60\text{kg} \cdot 9.81\text{m/s}^2)(1.5\text{m}) + (70\text{kg} \cdot 9.81\text{m/s}^2)(4.5\text{m}) + (120\text{N})(2.5\text{m}) \\ N_B(5\text{m}) &= 294.3\text{Nm} + 882.9\text{Nm} + 3090.15\text{Nm} + 300\text{Nm} \\ N_B(5\text{m}) &= 4567.35\text{Nm} \\ N_B &= \frac{4567.35\text{Nm}}{5\text{m}} \\ N_B &= 913.47\text{N} \\ N_B &= 913\text{N} \end{aligned}$$

$$\begin{aligned} \sum F_y = 0 &= N_A + N_B - F_c - F_M - F_f - F_g \\ N_A &= F_c + F_M + F_f + F_g - N_B \\ N_A &= (30\text{kg} \cdot 9.81\text{m/s}^2) + (60\text{kg} \cdot 9.81\text{m/s}^2) + (70\text{kg} \cdot 9.81\text{m/s}^2) + 120\text{N} - 913.47\text{N} \\ N_A &= 294.3\text{N} + 588.6\text{N} + 686.7\text{N} + 120\text{N} - 913.47\text{N} \\ N_A &= 776.13\text{N} \\ \underline{N_A} &= 776\text{N} \end{aligned}$$

## 6. Review

It is interesting that  $N_B$  is larger than  $N_A$ , because the weight of the mother and child combined (80 kg) is larger than that of the father (70 kg). However, when you sum the moments at point B instead of A, you get the same answer. The distance between the reaction forces and the nearest forces is important, as well as the magnitude of the forces themselves. The distance between A and  $F_c$  is 1 m, while the distance between B and  $F_f$  is only 0.5 m.

Additionally, it makes sense that both  $N_A$  and  $N_B$  are positive, i.e. are in the positive y direction.



## Example 3.6.2: Couples, Submitted by Kirsty MacLellan

### 1. Problem

A water valve is opened by a wheel with a diameter of 10 inches. It takes 7.5 lb of force to open the valve. What is the moment it takes to open the valve?

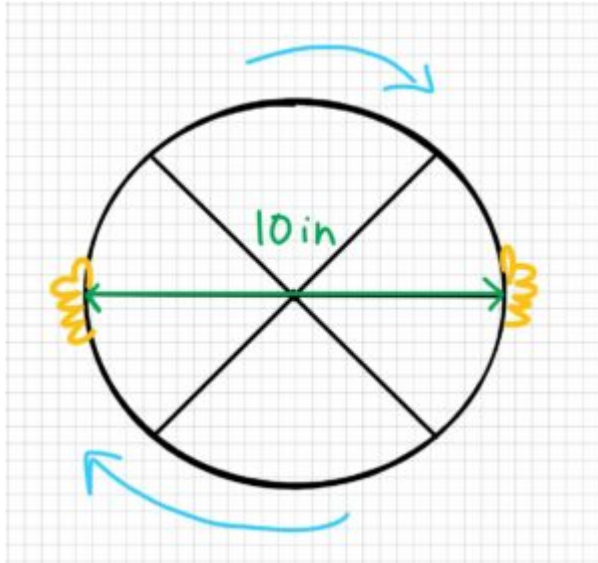
Real-life scenario:



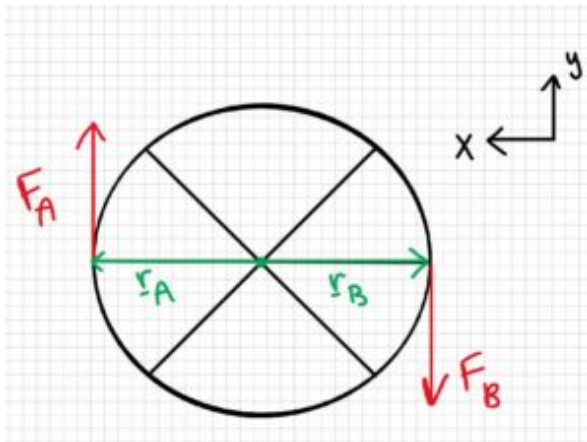
Source: <https://www.pxfuel.com/en/free-photo-ekahu>

### 2. Draw

Sketch:



Free-body diagram:



### 3. Knowns and Unknowns

Known:

$$d = 10 \text{ in}$$

$$F = 7.5 \text{ lb}$$

Unknown:  $M$

### 4. Approach

Determine the moment by finding the cross product of  $\underline{r}_A$  and  $\underline{F}_A$ , then  $\underline{r}_B$  and  $\underline{F}_B$ , then add.

### 5. Analysis

Find radius:

$$r = \frac{d}{2} = \frac{10 \text{ in}}{2} = 5 \text{ in} = 5 \text{ in} \times \frac{1 \text{ ft}}{12 \text{ in}} = 0.42 \text{ ft}$$

Find  $\underline{r}_A$ ,  $\underline{F}_A$ ,  $\underline{r}_B$ , and  $\underline{F}_B$  in vector form:

```


$$\underline{r}_A = \begin{bmatrix} 0.42 \\ 0 \\ 0 \end{bmatrix} \text{ ft}; \underline{F}_A = \begin{bmatrix} 0 \\ 7.5 \\ -0.42 \end{bmatrix} \text{ lb}$$


$$\underline{r}_B = \begin{bmatrix} 0 \\ 0 \\ -7.5 \end{bmatrix} \text{ ft}; \underline{F}_B = \begin{bmatrix} 0 \\ 0 \\ -7.5 \end{bmatrix} \text{ lb}$$


```

Find  $\underline{M}_A$ :

```


$$\underline{M}_A = \underline{r}_A \times \underline{F}_A = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0.42 & 0 & 0 \\ 0 & 7.5 & 0 \end{bmatrix}$$


$$= \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \begin{bmatrix} 0.42 & 0 & 0 \\ 0 & 7.5 & 0 \end{bmatrix}$$


$$= \begin{bmatrix} 0.42 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$


$$= \begin{bmatrix} 0.42 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$


$$\underline{M}_A = (\underline{\hat{i}}(0) - \underline{\hat{j}}(0) + \underline{\hat{k}}(0.42 \cdot 7.5 - 0)) \text{ ft} \cdot \text{lb}$$


$$\underline{M}_A = 3.15 \underline{\hat{k}} \text{ ft} \cdot \text{lb}$$


```

Find  $\underline{M}_B$ :

```


$$\underline{M}_B = \underline{r}_B \times \underline{F}_B = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -0.42 & 0 & 0 \\ 0 & -7.5 & 0 \end{bmatrix}$$


$$= \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \begin{bmatrix} -0.42 & 0 & 0 \\ 0 & -7.5 & 0 \end{bmatrix}$$


$$= \begin{bmatrix} -0.42 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$


$$= \begin{bmatrix} -0.42 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$


$$\underline{M}_B = (\underline{\hat{i}}(0) - \underline{\hat{j}}(0) + \underline{\hat{k}}(-0.42 \cdot -7.5 - 0)) \text{ ft} \cdot \text{lb}$$


$$\underline{M}_B = 3.15 \underline{\hat{k}} \text{ ft} \cdot \text{lb}$$


```

Add  $\underline{M}_A$  and  $\underline{M}_B$  to get  $\underline{M}$ :

$$\underline{M} = \underline{M}_A + \underline{M}_B \\ \underline{M} = 3.15\text{ft} \cdot \text{lb} + 3.15\text{ft} \cdot \text{lb} \\ \underline{M} = 6.3\text{ft} \cdot \text{lb}$$

### 6. Review

This answer makes sense because there is only moment acting in the k direction.

Note: we could have come to the same answer using the formula  $M = F \cdot d$ , which would have been faster.

$$M = F \cdot d \\ M = 7.5\text{lb} \cdot 10\text{in} \left( \frac{1\text{ft}}{12\text{in}} \right) \\ M = 7.5\text{lb} \cdot \frac{5}{6}\text{ft} \\ M = 6.25\text{ft} \cdot \text{lb}$$

This answer is slightly more accurate, because we didn't round when converting between inches and feet (in the original solution, we rounded 0.416667 to 0.42).

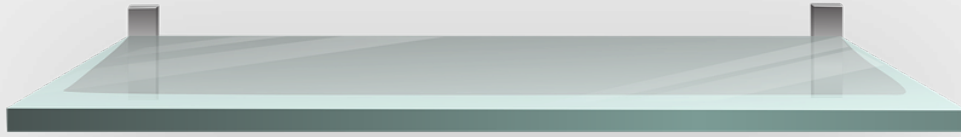
## Example 3.6.3: Distributed Load, Submitted by Luciana Davila

### 1. Problem

A shelf on the wall is 1.5 meters away from the floor. The shelf has a length of 100 cm. A person starts putting different objects on it to create a distributed load. The load created a curve described by:

$$w = 4x^4 + 2 \text{ N/m.}$$

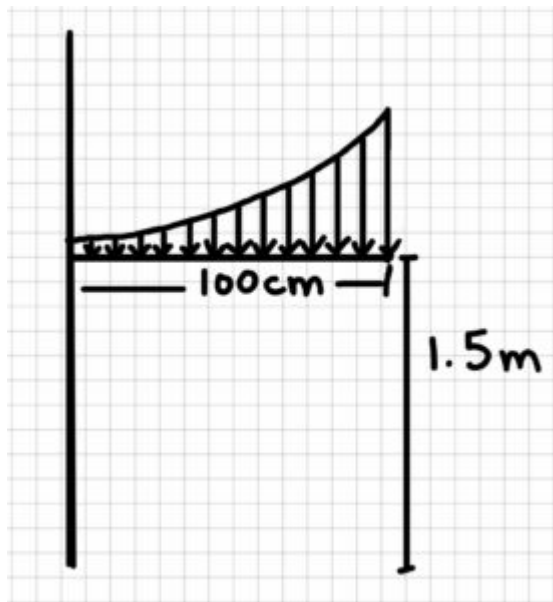
Calculate the resultant force and how far it is acting from the wall.



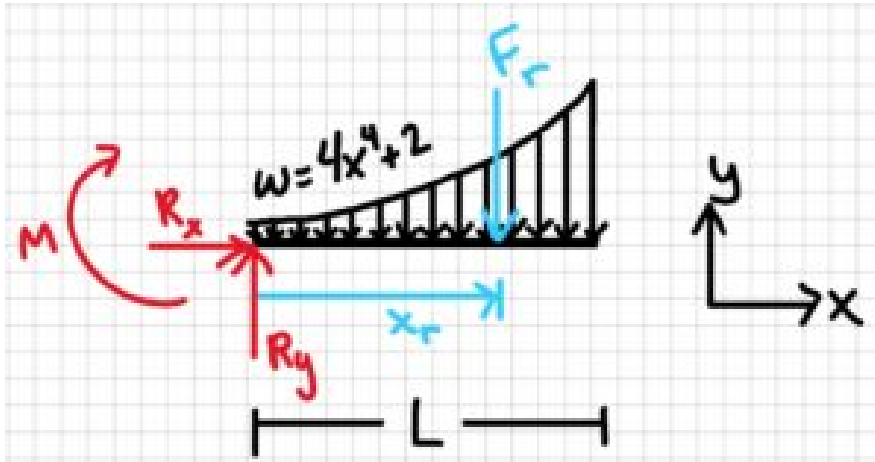
Source: <https://www.maxpixel.net/photo-576088>

## 2. Draw

Sketch:



Free-body diagram:



### 3. Knowns and Unknowns

Knowns:

- $w = 4x^4 + 2$  N/m
- $L = 100$  cm
- $x_{\min} = 0$
- $x_{\max} = 1$

Unknowns:  $x_r$ ,  $F_r$

### 4. Approach

Use distributed load equations:

$$F_r = \int_{x_{\min}}^{x_{\max}} w dx \quad X_r = \frac{\int_{x_{\min}}^{x_{\max}} x w(x) dx}{\int_{x_{\min}}^{x_{\max}} w dx}$$

### 5. Analysis

Solve for  $F_r$ :

$$F_r = \int_0^1 (4x^4 + 2) dx \quad N \quad F_r = \left( \frac{4x^5}{5} + 2x \right) \bigg|_0^1 \quad N \quad F_r = \left( \frac{4}{5} + 2 \right) N \quad F_r = 2.8 N$$

Solve for  $x_r$ :

$$X_r = \frac{\int_0^1 x(4x^4 + 2) dx}{\int_0^1 (4x^4 + 2) dx} \quad X_r = \frac{\int_0^1 (4x^5 + 2x) dx}{2.8 N} \\ X_r = \frac{\left( \frac{4x^6}{6} + \frac{2x^2}{2} \right) \bigg|_0^1}{2.8 N} \quad X_r = \frac{\left( \frac{2}{3} + 1 \right) N/m}{2.8 N} \quad X_r = 0.59 m$$

### 6. Review

The function shows an increasing curve on the interval, so it makes sense that the resultant force would be applied closer to the right end of the beam than the left end.

# CHAPTER 4: RIGID BODIES

This is arguably the most fundamental chapter for Statics. Learn these concepts and the next two chapters will make a lot of sense. Without this chapter, the next chapters will be much more confusing. When people talk about Statics, this chapter contains the concepts they are talking about. You will use free-body diagrams and the equilibrium equations in many other courses. Here are the sections in this Chapter:

- 4.1 External Forces - Types of external forces
- 4.2 Rigid Body Free Body Diagrams - How to model problems to be able to solve them **\*\* very important section**
- 4.3 Rigid Body Equilibrium Equations - How to apply what you learned on particles to rigid bodies
- 4.4 Friction and Impending Motion - Special cases of an external force looking at slipping and tipping
- 4.5 Examples - Examples from your peers

Here are the important equations for this chapter.

## Ch 4 Equations

### *FBD tips*

- |                 |               |  |
|-----------------|---------------|--|
| • Gravitational | $F_g = mg$    | Straight down at cg                      |
| • Normal        | $F_N$         | Perpendicular to plane                   |
| • Friction      | $F_f = \mu N$ | Parallel to plane                        |
| • Spring        | $F_s = -ks$   | Opposes motion (dynamic)                 |
| • Applied       | $F_A$         | At location applied (pulling or pushing) |

### Part FBD steps

1. Draw shape
2. Add coordinate frame
3. Replace forces with arrows
4. Label each force uniquely

\* If negative, in opposite direction as drawn

### System FBD steps

1. Draw original image.
2. Draw system FBD & coord frame
3. Draw each part separately & coord frame with equal and opposite arrows for internal system forces

## Equilibrium Equations

### *Particle*

$$\begin{aligned}\Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma F_z &= 0\end{aligned}$$

### *Rigid Body in 2D*

$$\begin{aligned}\Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma M_O &= 0\end{aligned}$$

### *Rigid Body in 3D*

$$\begin{aligned}\Sigma F_x &= 0 & \Sigma M_{x'} &= 0 \\ \Sigma F_y &= 0 & \Sigma M_{y'} &= 0 \\ \Sigma F_z &= 0 & \Sigma M_{z'} &= 0\end{aligned}$$

# 4.1 External Forces

When we say ‘forces’ in Statics, we are generally talking about external forces (such as the reaction forces discussed in the previous chapter) and internal forces (that we will discuss in Chapter 5 and 6). Generally, external forces include:

- gravitational force (or weight)
- normal force
- frictional force
- spring force
- applied force (such as reaction forces & tension) – this also includes applied moments such as from motors

In science class you probably learned about the fundamental forces of nature: gravitational, electromagnetic, and weak and strong nuclear forces. Normal force, friction, spring, and applied forces are all types of electromagnetic forces. The charged and neutral particles attract or repel each other. For example, the reason your laptop doesn’t fall through the table is that the electrons in the atoms of the two objects are repelling each other, and both objects are being pulled down by another fundamental force: gravitational force. See [this page](#) for more information. The four fundamental forces are beyond the scope of this Statics class, but it’s important to know the background how the external forces operate. In this class, we’ll use our understanding of the external forces to learn how to quantify the forces and calculate the value of other forces.

To calculate each force individually, use the following equations:

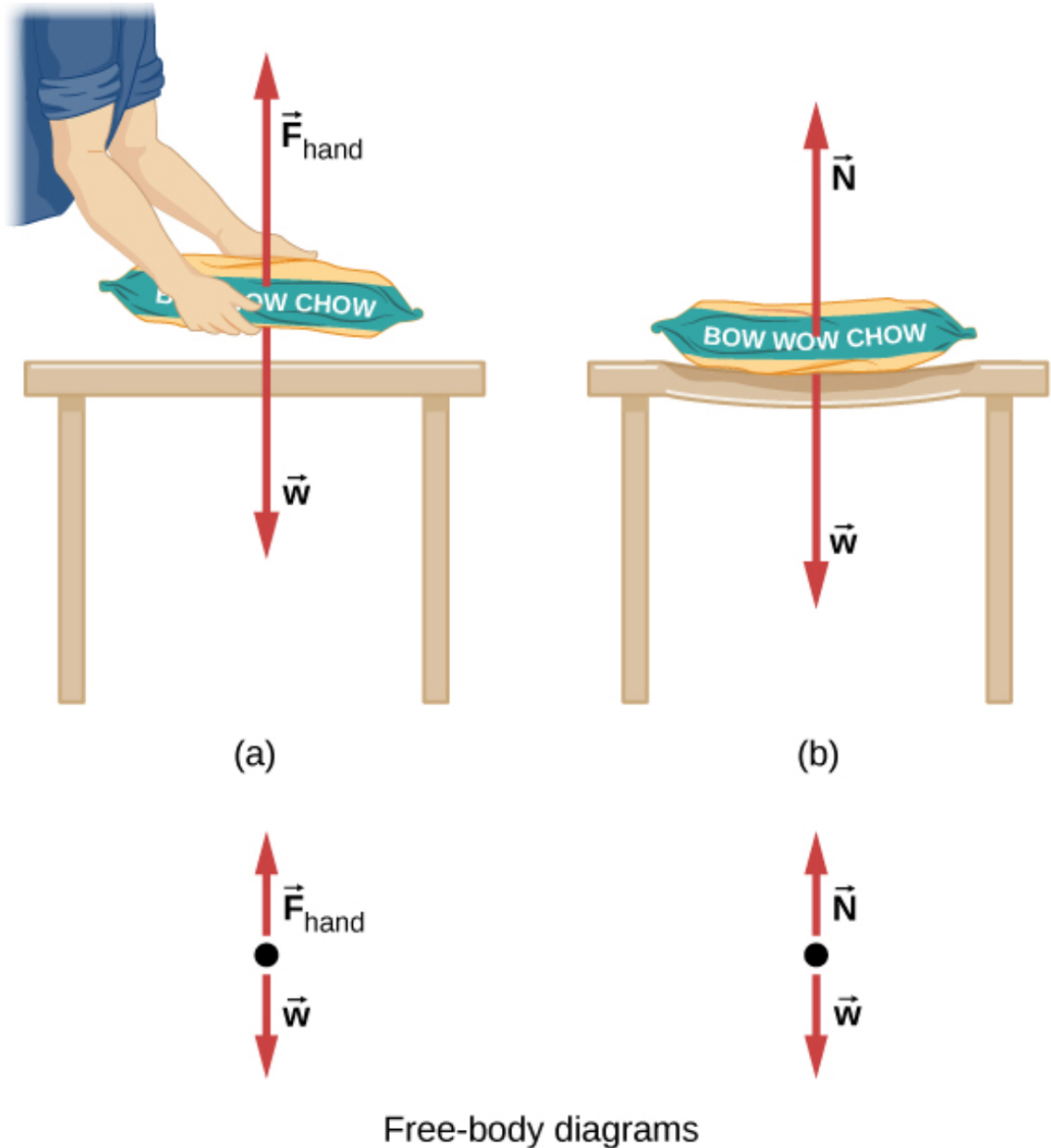
- Gravity:  $F_g = mg$
- Normal: Calculated
- Friction:  $F_f = \mu N$
- Spring:  $F_s = -kx$
- Applied: Measured or calculated

Forces are given many names, such as push, pull, thrust, and weight. Traditionally, forces have been grouped into several categories and given names relating to their source, how they are transmitted, or their effects. Several of these categories are discussed in this section.



## Normal Force

Weight (also called the force of gravity) is a pervasive force that acts at all times and must be counteracted to keep an object from falling. You must support the weight of a heavy object by pushing up on it when you hold it stationary. But how do inanimate objects like a table support the weight of a mass placed on them, such as shown in the figure below? When the bag of dog food is placed on the table, the table sags slightly under the load. This would be noticeable if the load were placed on a card table, but even a sturdy oak table deforms when a force is applied to it. Unless an object is deformed beyond its limit, it will exert a restoring force much like a deformed spring (or a trampoline or diving board). The greater the deformation, the greater the restoring force. Thus, when the load is placed on the table, the table sags until the restoring force becomes as large as the weight of the load. At this point, the net external force on the load is zero. That is the situation when the load is stationary on the table. The table sags quickly and the sag is slight, so we do not notice it. But it is similar to the sagging of a trampoline when you climb onto it.



**Figure 5.21** (a) The person holding the bag of dog food must supply an upward force  $\vec{F}_{\text{hand}}$  equal in magnitude and opposite in direction to the weight of the food  $\vec{w}$  so that it doesn't drop to the ground. (b) The card table sags when the dog food is placed on it, much like a stiff trampoline. Elastic restoring forces in the table grow as it sags until they supply a force  $\vec{N}$  equal in magnitude and opposite in direction to the weight of the load.

We must conclude that whatever supports a load, be it animate or not, must supply an upward force equal to the weight of the load, as we assumed in a few of the previous examples. If the force supporting the weight of an object, or a load, is perpendicular to the surface of contact between the load and its support, this force is defined as a **normal force** and here is given by the symbol  $\vec{N}$ .

$\vec{N}$ . (This is not the newton unit for force, N.) The word *normal* means perpendicular to a surface. This means that the normal force experienced by an object resting on a horizontal surface can be expressed in vector form as follows:

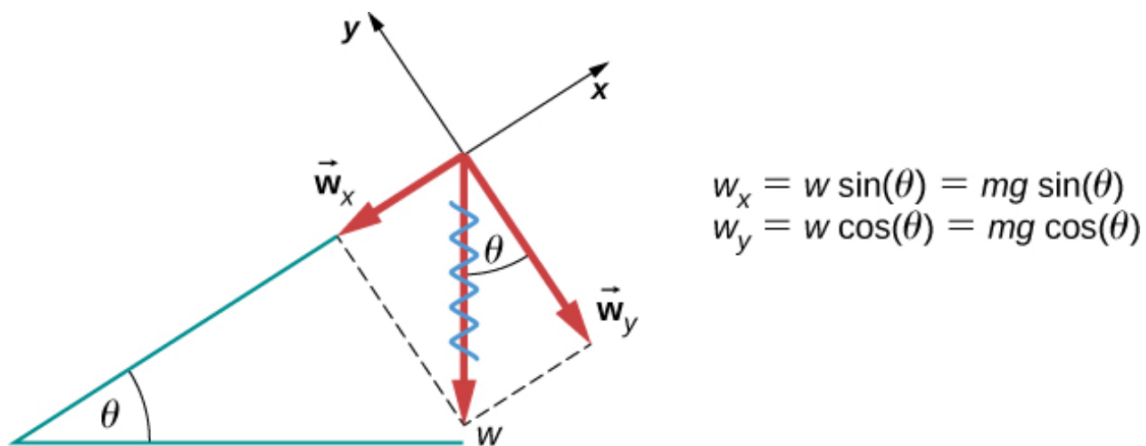
$$\vec{N} = -m\vec{g}$$

In scalar form, this becomes:

$$N = mg$$

The normal force can be less than the object's weight if the object is on an incline.

When an object rests on an incline that makes an angle  $\theta$  with the horizontal, the force of gravity acting on the object is divided into two components: a force acting perpendicular to the plane,  $w_y$ , and a force acting parallel to the plane,  $w_x$ . The normal force  $\vec{N}$  is typically equal in magnitude and opposite in direction to the perpendicular component of the weight  $w_y$ . The force acting parallel to the plane,  $w_x$ , causes the object to accelerate down the incline.



**Figure 5.23** An object rests on an incline that makes an angle  $\theta$  with the horizontal.

Be careful when resolving the weight of the object into components. If the incline is at an angle  $\theta$  with the horizontal, the weight vector  $\vec{w}$  is typically equal in magnitude and opposite in direction to the perpendicular component of the weight  $w_y$ . The force acting parallel to the plane,  $w_x$ , causes the object to accelerate down the incline.

none;max-height: none;min-width: 0px;min-height: 0px;color: #373d3f">to the horizontal, then the magnitudes of the weight components are:

$$w_x = w \sin \theta = mg \sin \theta$$

and

$$w_y = w \cos \theta = mg \cos \theta$$

We use the second equation to write the normal force experienced by an object resting on an inclined plane:

$$N = mg \cos \theta$$

Instead of memorizing these equations, it is helpful to be able to determine them from reason. To do this, we draw the right angle formed by the three weight vectors. The angle  $\theta$  of the incline is the same as the angle formed between  $w$  and  $w_y$ . Knowing this property, we can use trigonometry to determine the magnitude of the weight components:

$$\cos \theta = \frac{w_y}{w}, \therefore w_y = w \cos \theta = mg \cos \theta \quad \sin \theta = \frac{w_x}{w}, \therefore w_x = w \sin \theta = mg \sin \theta$$

## Tension

A **tension** is a force along the length of a medium; in particular, it is a pulling force that acts along a stretched flexible connector, such as a rope or cable. The word “tension” comes from a Latin word meaning “to stretch.” Not coincidentally, the flexible cords that carry muscle forces to other parts of the body are called *tendons*.

Any flexible connector, such as a string, rope, chain, wire, or cable, can only exert a pull parallel to its length; thus, a force carried by a flexible connector is a tension with a direction parallel to the connector. Tension is a pull in a connector. Consider the phrase: “You can’t push a rope.” Instead, tension force pulls outward along the two ends of a rope.

Consider a person holding a mass on a rope. If the 5.00-kg mass in the figure is stationary, then its acceleration is zero and the net force is zero. The only external forces acting on the mass are its weight and the tension supplied by the rope. Thus,

$$F_{\text{net}} = T - w = 0$$

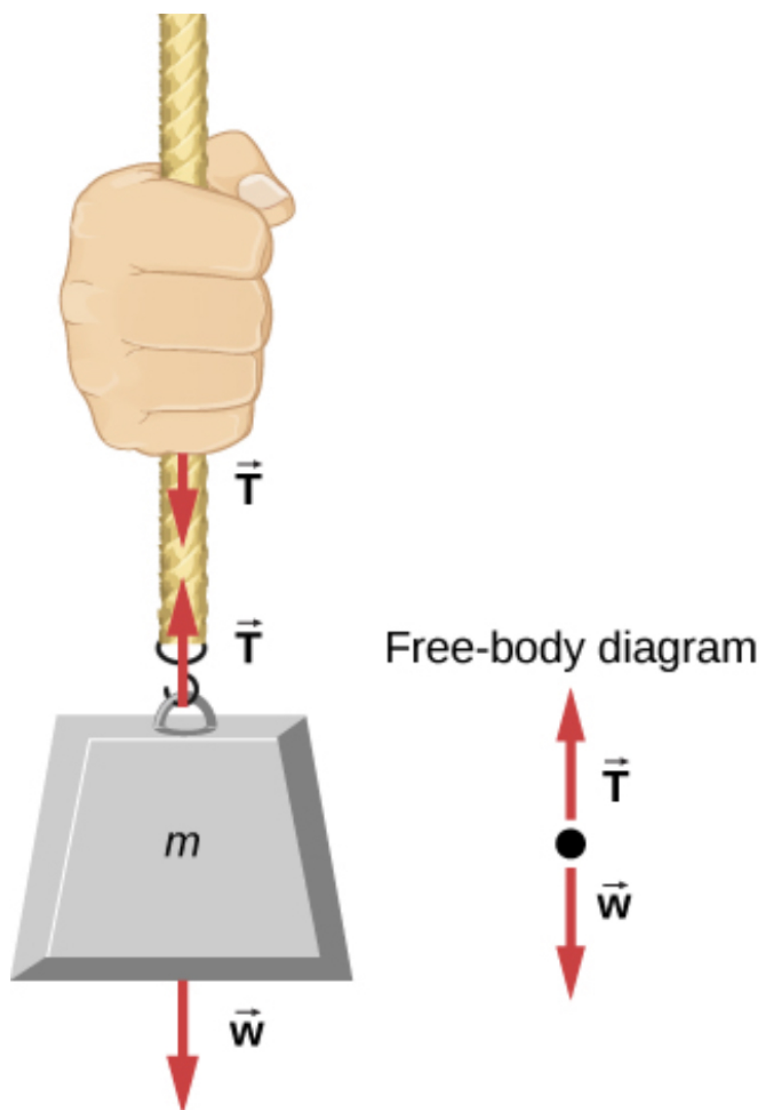
where  $T$  and  $w$  are the magnitudes of the tension and weight, respectively, and their signs indicate direction, with up being positive. As we proved using Newton’s second law, the tension equals the weight of the supported mass:

$$T = w = mg$$

Thus, for a 5.00-kg mass (neglecting the mass of the rope), we see that

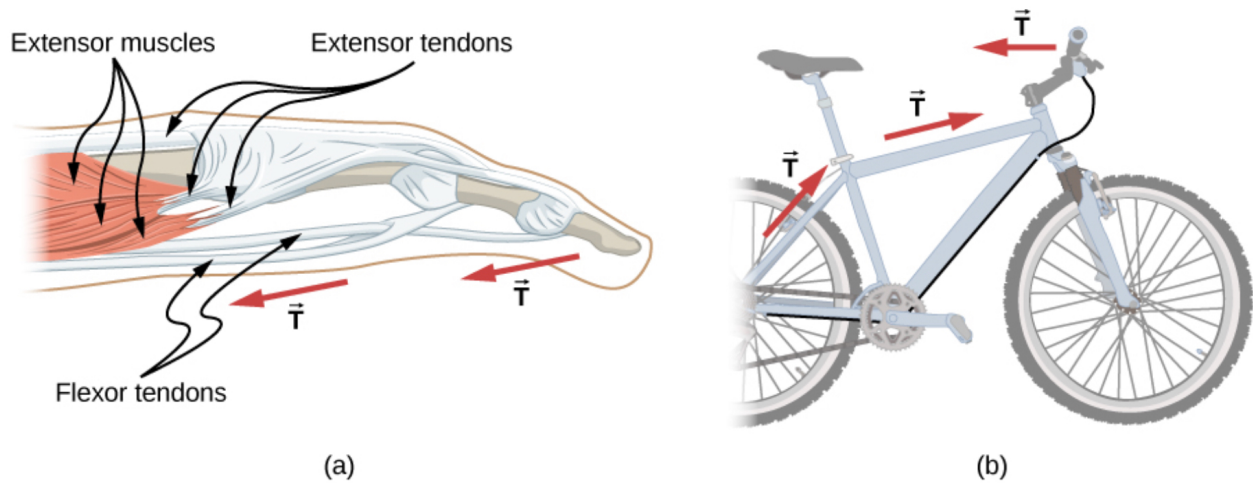
$$T = mg = (5.00 \text{ kg})(9.80 \text{ m/s}^2) = 49.0 \text{ N}$$

If we cut the rope and insert a spring, the spring would extend a length corresponding to a force of 49.0 N, providing a direct observation and measure of the tension force in the rope.



**Figure 5.24** When a perfectly flexible connector (one requiring no force to bend it) such as this rope transmits a force  $\vec{T}$ , that force must be parallel to the length of the rope, as shown. By Newton's third law, the rope pulls with equal force but in opposite directions on the hand and the supported mass (neglecting the weight of the rope). The rope is the medium that carries the equal and opposite forces between the two objects. The tension anywhere in the rope between the hand and the mass is equal. Once you have determined the tension in one location, you have determined the tension at all locations along the rope.

Flexible connectors are often used to transmit forces around corners, such as in a hospital traction system, a tendon, or a bicycle brake cable. If there is no friction, the tension transmission is undiminished; only its direction changes, and it is always parallel to the flexible connector, as shown below:



**Figure 5.25** (a) Tendons in the finger carry force  $T$  from the muscles to other parts of the finger, usually changing the force's direction but not its magnitude (the tendons are relatively friction free). (b) The brake cable on a bicycle carries the tension  $T$  from the brake lever on the handlebars to the brake mechanism. Again, the direction but not the magnitude of  $T$  is changed.

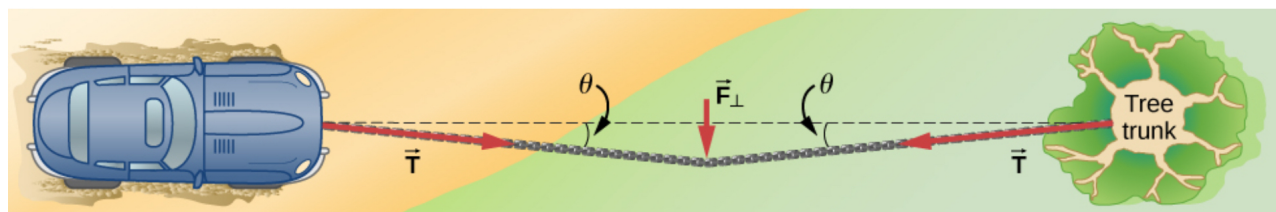
If we wish to create a large tension, all we have to do is exert a force perpendicular to a taut flexible connector. We can see that the tension in the rope is related to the force acting perpendicularly in the following way:

$$T = \frac{w}{2 \sin \theta}$$

We can extend this expression to describe the tension  $T$  created when a perpendicular force ( $F_{\perp}$ ) is exerted at the middle of a flexible connector:

$$T = \frac{F_{\perp}}{2 \sin \theta}$$

The angle between the horizontal and the bent connector is represented by  $\theta$ . In this case,  $T$  becomes large as  $\theta$  approaches zero. Even the relatively small weight of any flexible connector will cause it to sag, since an infinite tension would result if it were horizontal (i.e.,  $\theta=0$  and  $\sin \theta=0$ ). For example, the image below shows a situation where we wish to pull a car out of the mud when no tow truck is available. Each time the car moves forward, the chain is tightened to keep it as straight as possible. The tension in the chain is given by  $T = \frac{F_{\perp}}{2 \sin \theta}$  and since  $\theta$  is small,  $T$  is large. This situation is analogous to the tightrope walker, except that the tensions shown here are those transmitted to the car and the tree rather than those acting at the point where  $F_{\perp}$  is applied.



**Figure 5.28** We can create a large tension in the chain—and potentially a big mess—by pushing on it perpendicular to its length, as shown.

## Friction

Friction is a resistive force opposing motion or its tendency. Imagine an object at rest on a horizontal surface. The net force acting on the object must be zero, leading to equality of the weight and the normal force, which act in

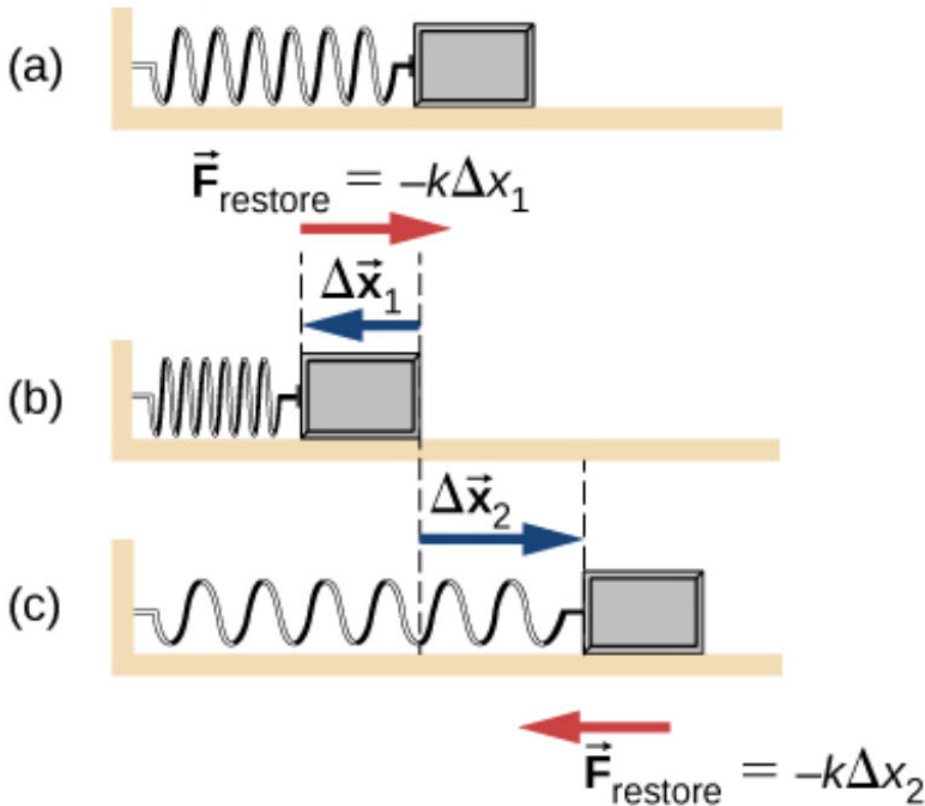
opposite directions. If the surface is tilted, the normal force balances the component of the weight perpendicular to the surface. If the object does not slide downward, the component of the weight parallel to the inclined plane is balanced by friction. Friction is discussed in greater detail in the next chapter.

## Spring Force

A spring is a special medium with a specific atomic structure that has the ability to restore its shape, if deformed. To restore its shape, a spring exerts a restoring force that is proportional to and in the opposite direction in which it is stretched or compressed. This is the statement of a law known as Hooke's law, which has the mathematical form

$$\vec{F} = -k\vec{x}$$

The constant of proportionality  $k$  is a measure of the spring's stiffness. The line of action of this force is parallel to the spring axis, and the sense of the force is in the opposite direction of the displacement vector. The displacement must be measured from the relaxed position;  $x=0$  when the spring is relaxed.



**Figure 5.29** A spring exerts its force proportional to a displacement, whether it is compressed or stretched. (a) The spring is in a relaxed position and exerts no force on the block. (b) The spring is compressed by displacement  $\Delta \vec{x}_1$  of the object and exerts restoring force  $-k\Delta \vec{x}_1$ . (c) The spring is stretched by displacement  $\Delta \vec{x}_2$  of the object and exerts restoring force  $-k\Delta \vec{x}_2$ .

Source: University Physics Volume 1, Openstax CNX.

<https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/5-6-common-forces/>

### Key Takeaways

- Basically: External forces include: gravitational, applied, normal, frictional, and spring.
- Application: Everything. A book on a table, Tigger bouncing on his tail, a shooting star, and a soccer ball rolling into the goal.
- Looking Ahead: Ch 5 and 6 will look at internal forces. Section 4.3 will use the known forces to calculate the unknown forces. Section 4.2 will model the forces on a diagram.



## 4.2 Rigid Body Free Body Diagrams

Following what we learned in Section 2.2 on particle Free-Body Diagrams (FBDs), this section will expand on that for rigid bodies. The biggest difference between a particle and rigid body FBD is *where* the force is applied. In a rigid body FBD, you have to be precise about pointing the head of the force arrow to the location where it applied. For example, if we wanted to make a FBD of you and me high-5'ing, you would apply the force from your hand onto my hand, not at my center of mass.

In this section, first we will learn how to do a FBD for a part, then we look at how to model a system of multiple objects.

### 4.2.1 Part FBD

When modelling a single object using an FBD, you are simplifying a complex problem into specific forces using arrows and an object floating in space. The floor becomes a normal force arrow and a frictional force arrow. Pushing or pulling on an object becomes an applied force with the arrow pointing to or from (pushing or pulling) the location where the pushing or pulling occurs. Remember the rules from section 2.2 still apply:

- Add coordinate frame (*which way is positive  $x$  and positive  $y$ ?*)
- Replace surfaces with forces (*floor, hand, and objects touching it become arrows*)
- Point forces in the correct direction (*the head of the arrow points to where the force acts.  $F_G$  acts down*)
- Use unique (different) names (*be sure to name each force with a different name*).

Here are some tips to keep in mind about each of the forces:

- Gravity acts on every particle in an object. Because we don't want a million little arrows on the object, we sum the effect of gravity at the center of mass. This is also because we generally know the total mass of something and where that occurs on an object (often at the geometric center), so we concentrate the force of gravity at this center of mass.
- Normal forces always act *perpendicular to the surface*, so if the ground is at an angle, then the normal force acts 90 degrees from that angle (perpendicular).
- Frictional forces act *parallel to the plane* between the two surfaces. This makes it a shear force, which we'll look at in Chapter 6.

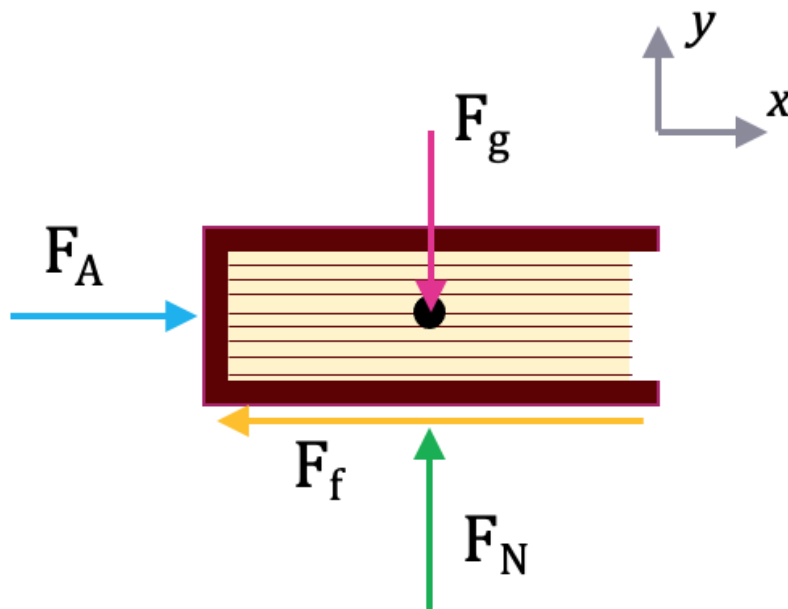
- Friction always *opposes motion*, a fact that will be very important in your dynamics class.
- Spring force is often shown as negative because the force acts in the opposite direction of the motion traveled. In application, you set the direction of the frictional force to match if it is pushing or pulling.
- Applied forces (and moments), such as distributed loads, motors, pushing on an object, tension, etc.

The steps to make a FBD are:

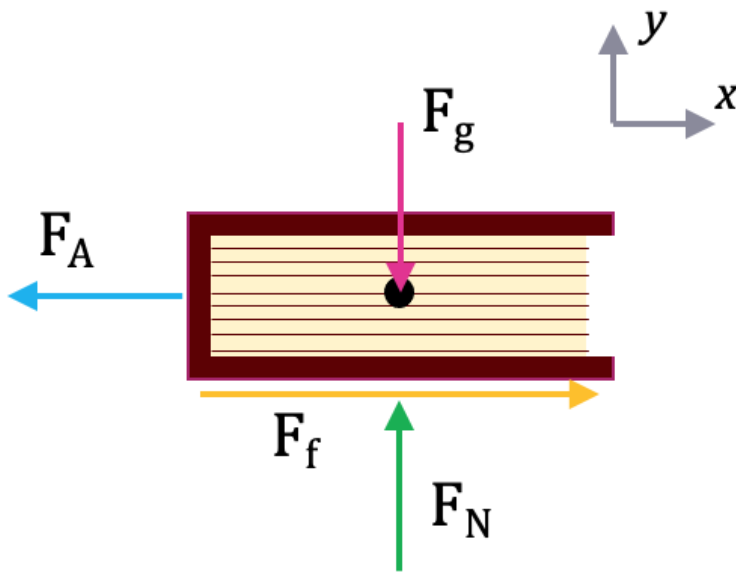
1. Draw shape
2. Add coordinate frame
3. Replace forces with arrows
4. Label each force uniquely

To model a book being pushed across the table, you would apply the following forces at the following locations (see image below)

- the normal force on the bottom of the book (green arrow)
- the frictional force running along the bottom surface between the book and table (yellow arrow)
- the gravitational force acting at the center of mass (pink arrow)
- any applied force at the point of application, such as your hand pushing on the book (blue arrow)

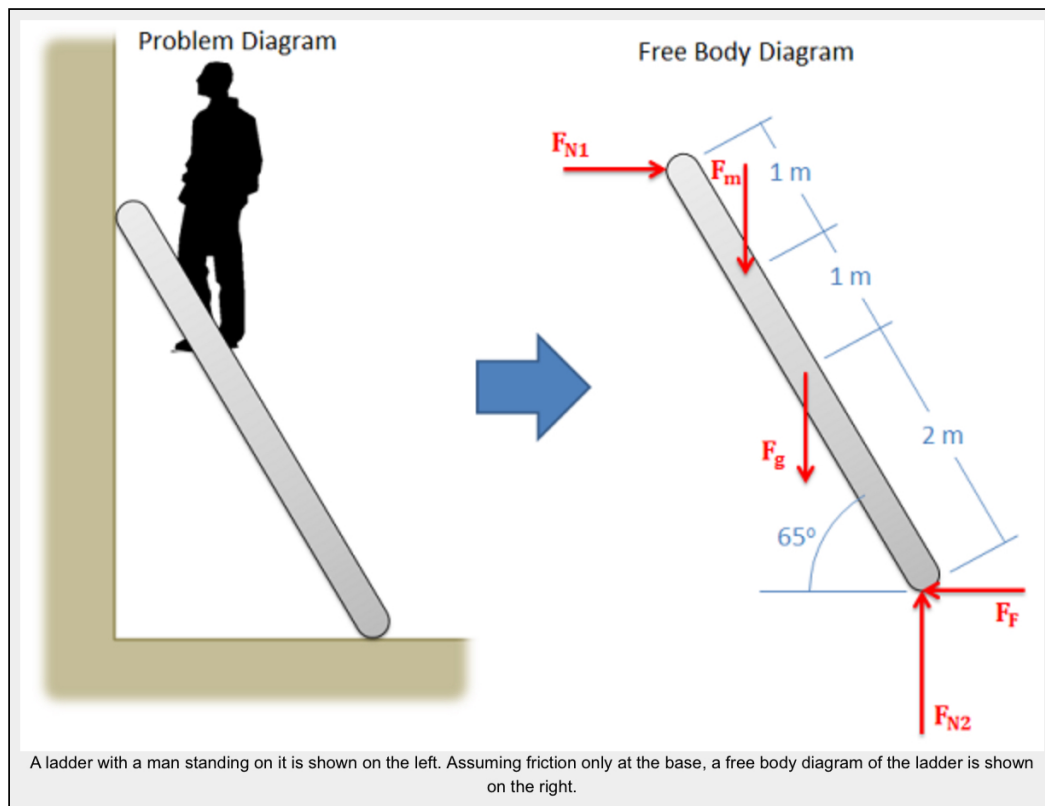


If instead, the book were being pulled by a string, the image would be the same but the applied force and frictional force would change direction (because friction always opposes motion).



A free body diagram is a tool used to solve engineering mechanics problems. As the name suggests, the purpose of the diagram is to “free” the body from all other objects and surfaces around it so that it can be studied in isolation. We will also draw in any forces or moments acting on the body, including those forces and moments exerted by the surrounding bodies and surfaces that we removed.

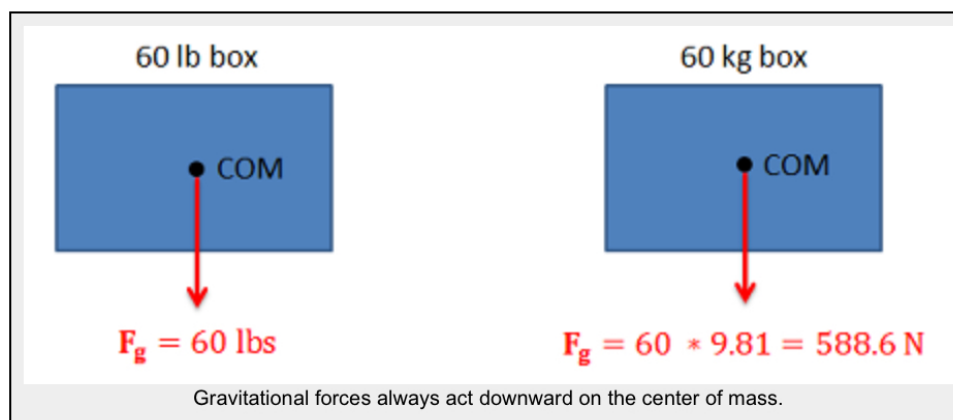
The diagram below shows a ladder supporting a person and the free body diagram of that ladder. As you can see, the ladder is separated from all other objects and all forces acting on the ladder are drawn in with key dimensions and angles shown.



The first step in solving most mechanics problems will be to construct a free body diagram. This simplified diagram will allow us to more easily write out the equilibrium equations for statics or strengths of materials problems, or the equations of motion for dynamics problems.

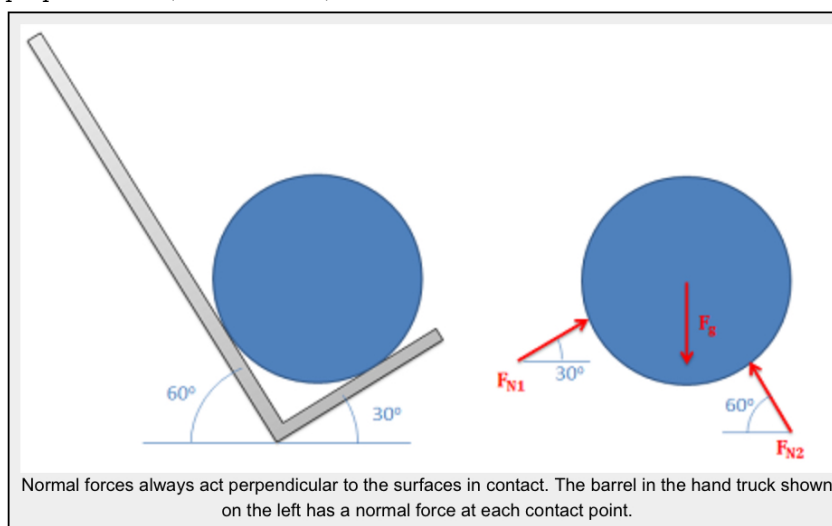
To construct the diagram we will use the following process.

1. First draw the body being analyzed, separated from all other surrounding bodies and surfaces. Pay close attention to the boundary, identifying what is part of the body, and what is part of the surroundings.
2. Second, draw in all **external** forces and moments acting directly on the body. Do not include any forces or moments that do not directly act on the body being analyzed. Do not include any forces that are **internal** to the body being analyzed. Some common types of forces seen in mechanics problems are:
  - **Gravitational Forces:** Unless otherwise noted, the mass of an object will result in a gravitational weight force applied to that body. This weight is usually given in pounds in the English system, and is modeled as 9.81 (g) times the mass of the body in kilograms for the metric system (resulting in a weight in Newtons). This force will always point down towards the center of the earth and act on the center of mass of the body.

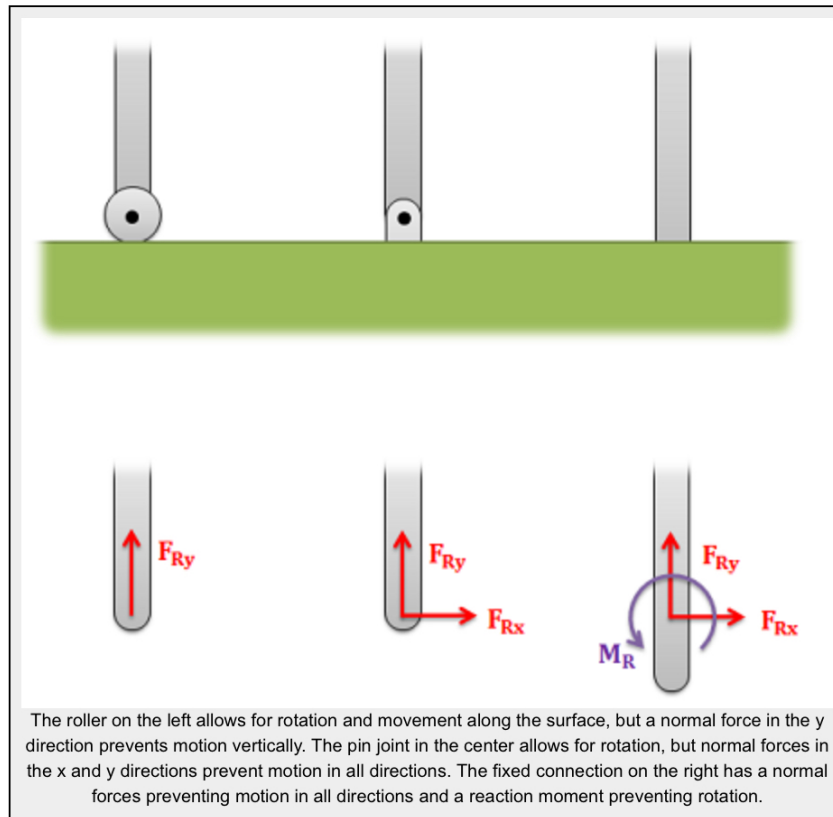


■ **Normal Forces (or Reaction Forces):** Every object in direct contact with the body will exert a normal force on that body which prevents the two objects from occupying the same space at the same time. Note that only objects in direct contact can exert normal forces on the body.

- An object in contact with another object or surface will experience a normal force that is perpendicular (hence normal) to the surfaces in contact.

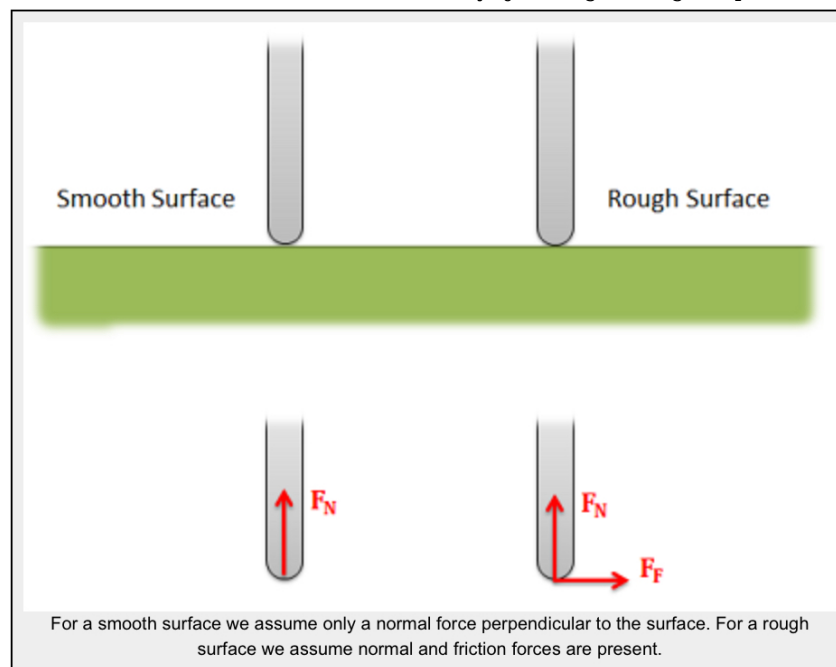


- Joints or connections between bodies can also cause reaction forces or moments, and we will have one force or moment for each type of motion or rotation the connection prevents.



■ **Friction Forces:** Objects in direct contact with the body can also exert friction forces on the body, which will resist the two bodies sliding against one another. These forces will always be perpendicular to the surfaces in contact. Friction is the subject of an entire chapter in this book, but for simple scenarios we usually assume rough or smooth surfaces.

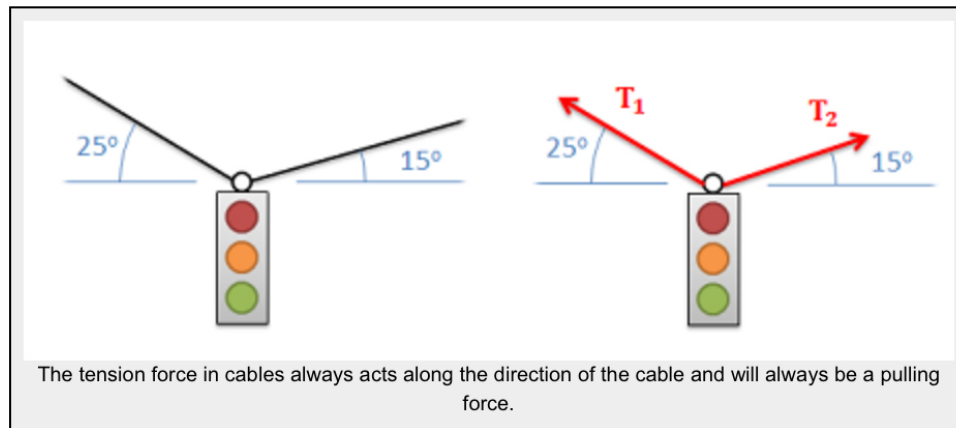
- For smooth surfaces we assume that there is no friction force.
- For rough surfaces we assume that the bodies will not slide relative to one another no matter what. In this case the friction force is always just large enough to prevent this



sliding.

■ **Tension in Cables:** Cables, wires or ropes attached to the body will exert a tension force on the body in the direction of the cable. These forces will always pull on the body, as ropes, cables

and other flexible tethers cannot be used for pushing.



- The above forces are the most common, but other forces such as pressure from fluids, spring forces and magnetic forces may exist and may act on the body.

3. Once the forces are identified and added to the free body diagram, the last step is to label any key dimensions and angles on the diagram.

Source: Engineering Mechanics, Jacob Moore, et al.

[http://wwwhttp://mechanicsmap.psu.edu/websites/1\\_mechanics\\_basics/1-6\\_free\\_body\\_diagrams/free\\_body\\_diagrams.h  
tmlw.oercommons.org/courses/mechanics-map-open-mechanics-textbook/view](http://wwwhttp://mechanicsmap.psu.edu/websites/1_mechanics_basics/1-6_free_body_diagrams/free_body_diagrams.htmlw.oercommons.org/courses/mechanics-map-open-mechanics-textbook/view)

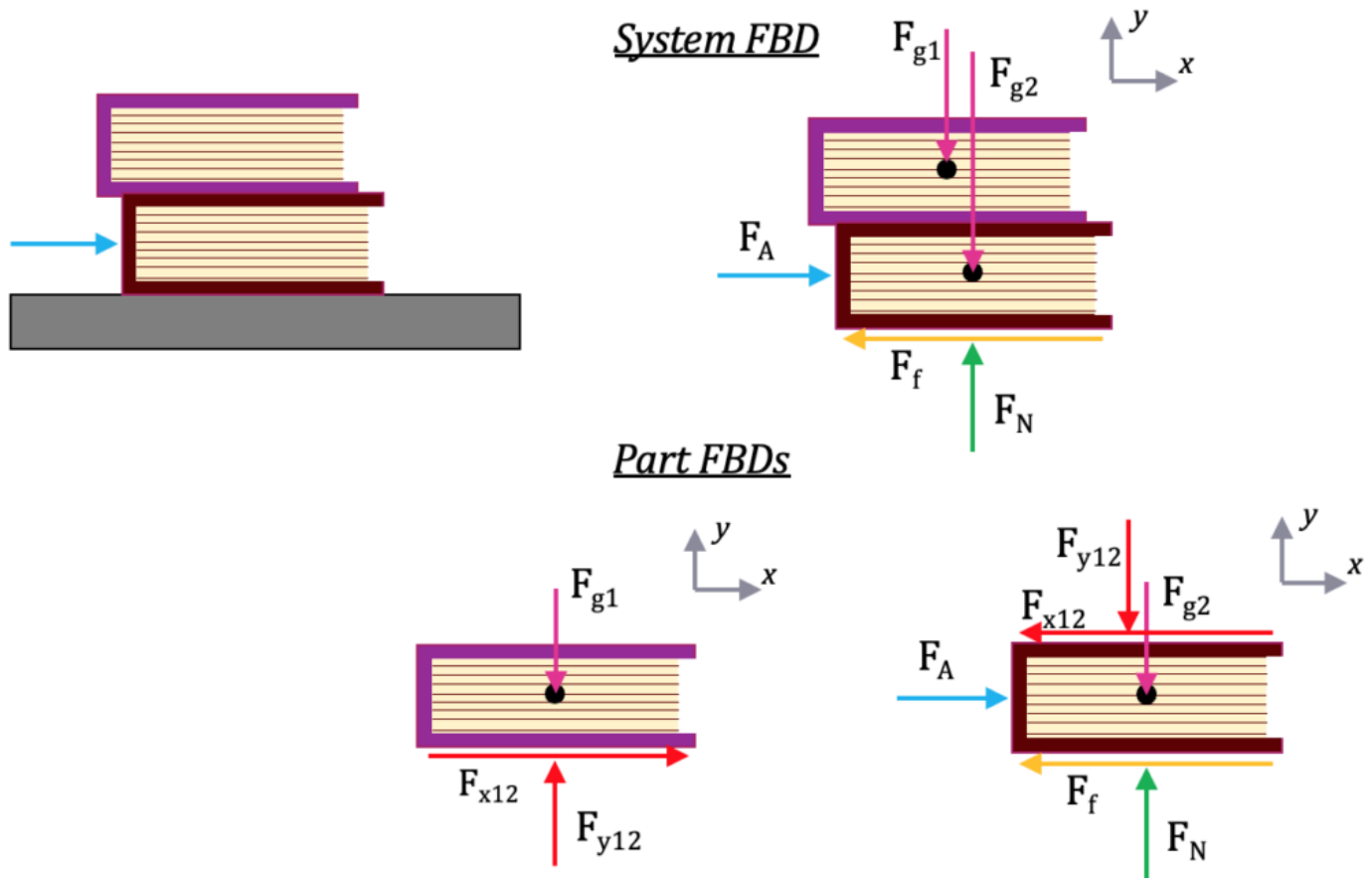
## 4.2.2 System FBD

A system free-body diagram is composed of multiple parts, so you can have multiple 'levels' to consider: the system level with all objects on the same FBD, and a part FBD for each individual part. This is especially helpful if you have a more unknowns than equations when using the equilibrium equations, so you can find more information by splitting the system up into individual parts.

For the system FBD, you look at the parts combined together and add only the external forces (gravity, applied, normal, frictional, spring). When you look at each part separately, you now have to include the interaction between the objects, replacing a part with forces (generally 2 forces: vertical and horizontal force).

For example, if there are 2 books stacked on top of each other, you now need 3 FBDs:

1. a system level FBD with both books,
2. a part FBD for the bottom book with the top book replaced by arrows (forces)
3. a part FBD for the top book with the bottom book replaced by arrows (forces)



To make a system FBD:

1. Draw system FBD using unique consistent labels (ie. a letter or number per part)
  - The system should be floating in space with no surface (such as the floor)
  - Include coordinate frame
  - Use only external forces on system FBD (gravity, applied, normal, frictional, spring).
  - DO NOT include internal forces
  - It is especially important to use unique labels, so the top book forces are labelled 1, and the bottom book forces are labelled 2 (or T for top and B for bottom, or A and B).
2. Draw a FBD for each part separately & coord frame with equal and opposite arrows for internal system forces
  - The part should be floating in space with no surfaces or other objects
  - Include a coordinate frame (yes, again! This is to ensure you didn't rotate the object).
  - Copy the external forces onto the part FBD from the system FBD with the identical labels and



arrow directions

- Now add internal forces replacing the other object with force arrows (red arrows)
- When you draw the second part FBD, follow the above bullets for the second object (with label 2 instead of 1, copying the system level external force labels). Note though that you use the **same labels for the internal forces** from the first part FBD, but the direction is reversed (left becomes right and up becomes down). Following Newton's laws, the objects exert equal and opposite forces on each other, and should cancel out at the system level, so they have the same label (magnitude) and opposite directions.

Some tips:

- Differentiate one object from the other through the labels, using either a letter or number for each part. These same labels go on the system level FBD (except for the internal labels).
- Use the same labels between the part and system FBDs for external forces. Don't change the label – that will make the equations impossible to solve and look like you have more unknowns.
- Use the same unique labels between the internal forces for the part FBDs, but in the opposite direction.
- If you know the location of the center of mass, you could combine the gravitational forces into 1 system level gravitational force. You could also model the gravitational force into one force per part acting at the center of mass for each object. This is the better method if you have to separate the objects to do the calculations.

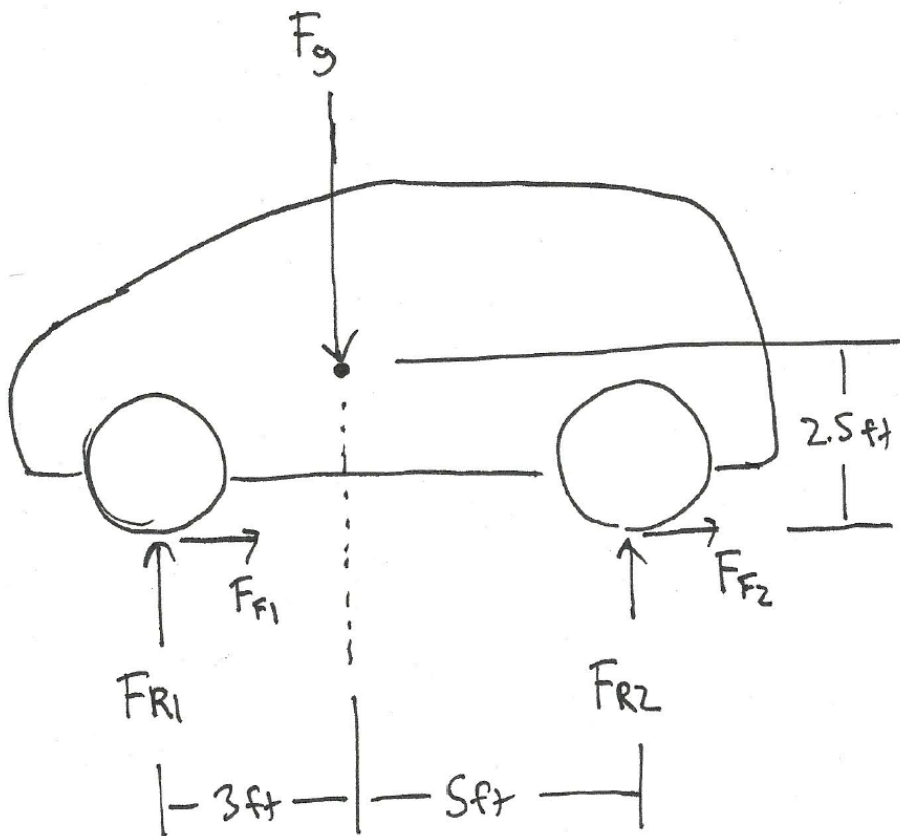
## 4.2.3 Examples

Here are some examples from:  
[http://mechanicsmap.psu.edu/websites/1\\_mechanics\\_basics/1-6\\_free\\_body\\_diagrams/free\\_body\\_diagrams.html](http://mechanicsmap.psu.edu/websites/1_mechanics_basics/1-6_free_body_diagrams/free_body_diagrams.html)

### Example 1: Part FBD

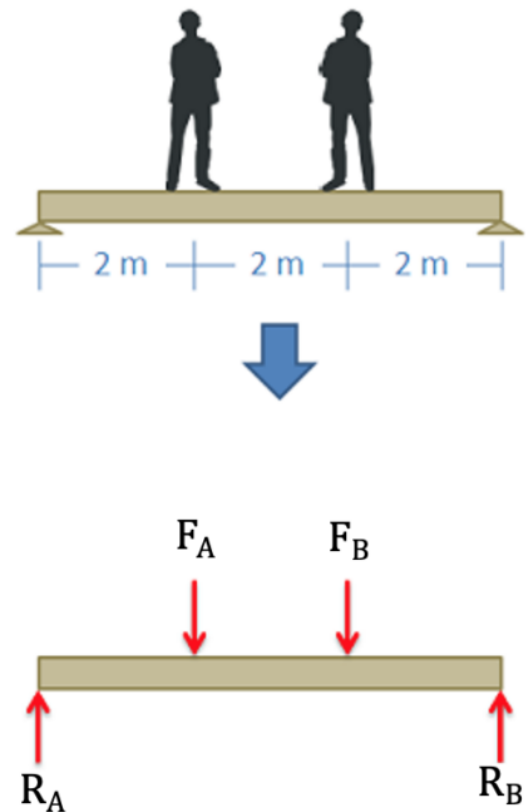
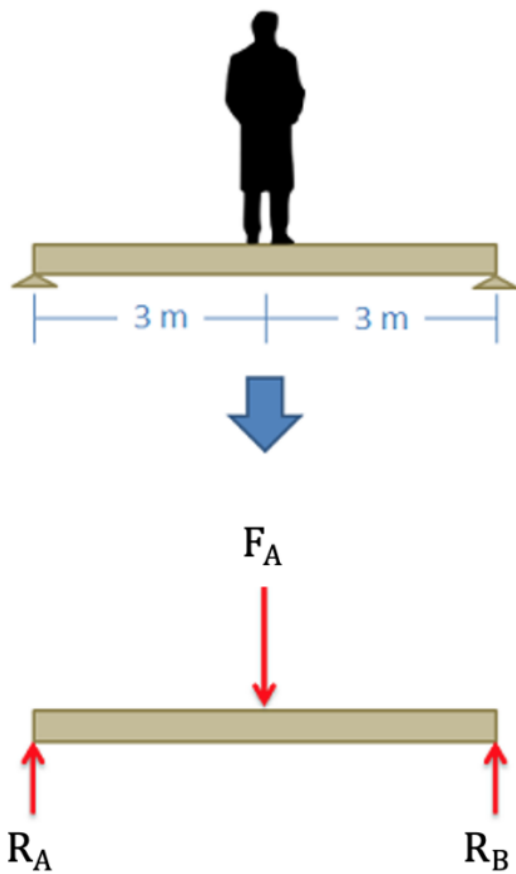
The car shown below is moving and then slams on the brakes locking up all four wheels. The distance between the two wheels is 8 feet and the center of mass is 3 feet behind and 2.5 feet above the point of contact between the

front wheel and the ground. Draw a free body diagram of the car as it comes to a stop.



Source: Engineering Mechanics, Jacob Moore et al.,  
[http://mechanicsmap.psu.edu/websites/1\\_mechanics\\_basics/1-6\\_free\\_body\\_diagrams/pdf/P3.pdf](http://mechanicsmap.psu.edu/websites/1_mechanics_basics/1-6_free_body_diagrams/pdf/P3.pdf)

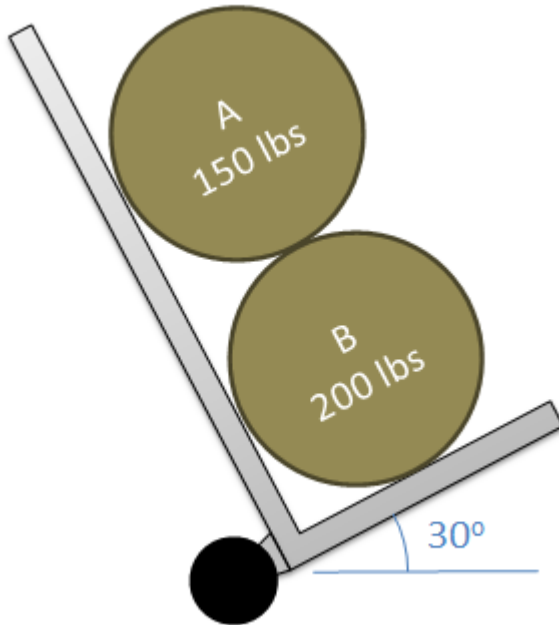
## Example 2: Part FBD (a beam)



Imaged adapted. Source: Engineering Mechanics, Jacob Moore, et al.  
[http://mechanicsmap.psu.edu/websites/4\\_statically\\_equivalent\\_systems/4-1\\_statically\\_equivalent\\_systems/images/equivalentexample.png](http://mechanicsmap.psu.edu/websites/4_statically_equivalent_systems/4-1_statically_equivalent_systems/images/equivalentexample.png)

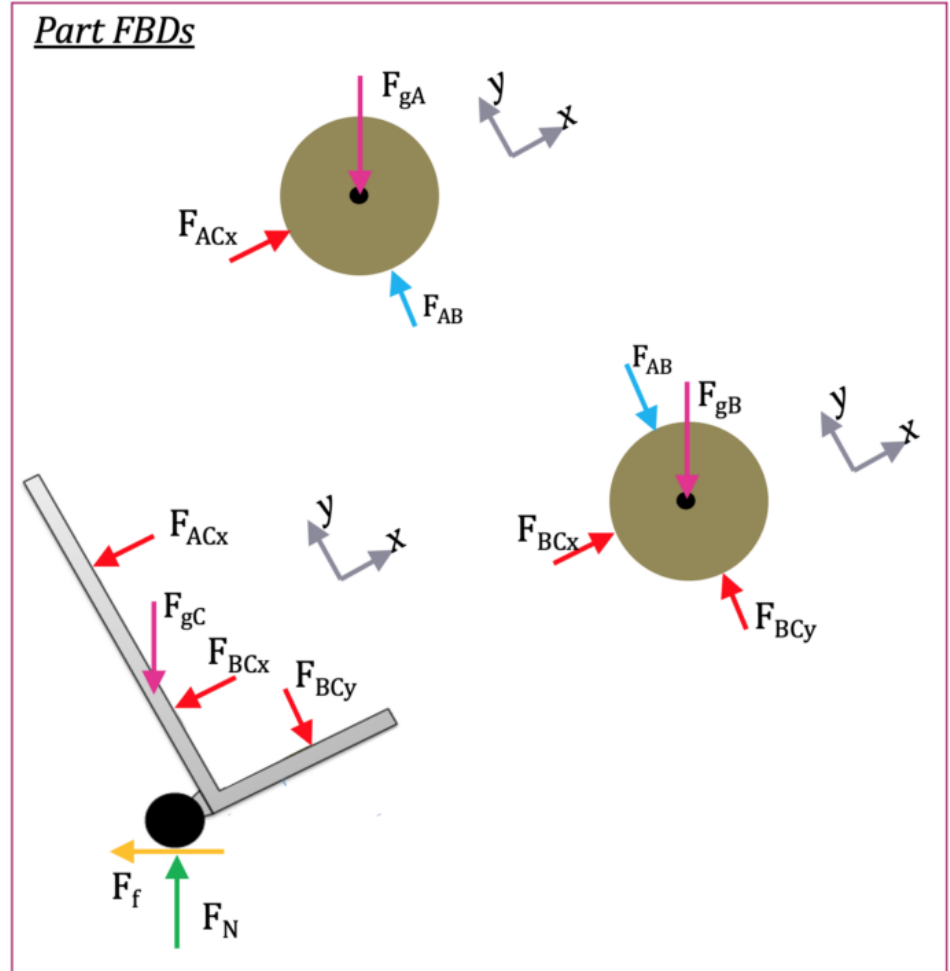
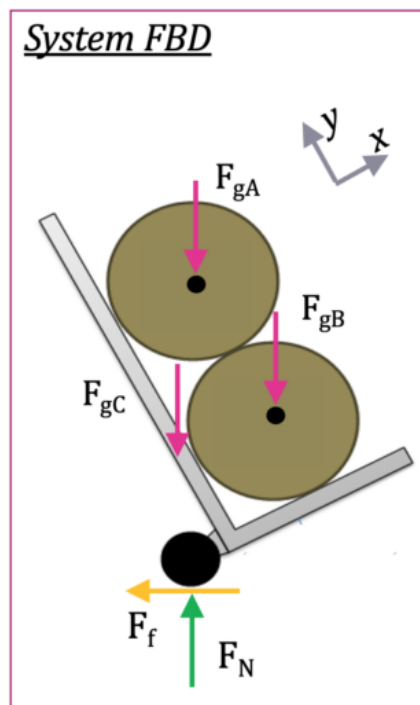
## Example 3: System FBD

Two equally sized barrels are being transported in a hand truck as shown below. Draw a free body diagram of each of the two barrels.



Source: Engineering Mechanics, Jacob Moore et al.,  
[http://mechanicsmap.psu.edu/websites/1\\_mechanics\\_basics/1-6\\_free\\_body\\_diagrams/pdf/P2.pdf](http://mechanicsmap.psu.edu/websites/1_mechanics_basics/1-6_free_body_diagrams/pdf/P2.pdf)

External forces in green, pink, and yellow. Internal forces between the cart and barrels (C and A/B) in red and between the barrels (A & B) in blue. Notice the matching labels for internal forces but opposing directions. Notice that the coordinate frame has been rotated consistently in all of the FBDs.



### Key Takeaways

*Basically:* A part free-body diagrams (FBDs) give you a way to model complicated problem in a simple way with arrows. Systems FBDs allows you to combine objects and analyze them separately.

*Application:* A bat swinging could be modeled as a part FBD with gravity and multiple applied forces (hands on one end and the ball on the other). You could model the moment the bat and ball are touching using a system FBD.

*Looking ahead:* You'll use a FBD in every step 2 in nearly every homework problem. These are especially helpful with Equilibrium Equations in the next section.

## 4.3 Rigid Body Equilibrium Equations

We use the equilibrium equations to calculate any unknown forces & moments using the known forces and values, and the following equations:

The particle equilibrium equations were covered in section 2.3. These are:

$$\begin{aligned} & \Sigma F_x = 0, \Sigma F_y = 0, \Sigma F_z = 0 \end{aligned}$$

Now for a rigid body where forces are analyzed at different points on a body, we can take moments into account. There are 3 equations for 2d and 4 equations for 3d:

Rigid Body-Two Dimensions

$$\begin{aligned} & \Sigma F_x = 0, \Sigma F_y = 0, \Sigma M_O = 0 \end{aligned}$$

Rigid Body-Three Dimensions

$$\begin{aligned} & \Sigma F_x = 0, \Sigma F_y = 0, \Sigma F_z = 0 \\ & \Sigma M_{x'} = 0, \Sigma M_{y'} = 0, \Sigma M_{z'} = 0 \end{aligned}$$

Because these are static bodies, the right side of the equations equal 0. In dynamics, they will equal the mass times the acceleration for translation and rotation.

For a **rigid body** in static equilibrium, that is a non-deformable body where forces are not concurrent, the sum of both the **forces** and the **moments** acting on the body must be equal to zero. The addition of moments (as opposed to particles where we only looked at the forces) adds another set of possible equilibrium equations, allowing us to solve for more unknowns as compared to particle problems.

Moments, like forces, are vectors. This means that our vector equation needs to be broken down into scalar components before we can solve the equilibrium equations. In a two dimensional problem, the body can only have clockwise or counter clockwise rotation (corresponding to rotations about the z axis). This means that a rigid body in a two dimensional problem has three possible equilibrium equations; that is, the sum of force components in the x

and y directions, and the moments about the z axis. The sum of each of these will be equal to zero.

For a two dimensional problem, we break our one vector force equation into two scalar component equations.

$$\sum \vec{F} = 0 \implies \sum F_x = 0; \sum F_y = 0$$

The one moment vector equation becomes a single moment scalar equation.

$$\sum \vec{M} = 0 \implies \sum M_z = 0$$

If we look at a three dimensional problem we will increase the number of possible equilibrium equations to six. There are three equilibrium equations for force, where the sum of the components in the x, y, and z direction must be equal to zero. The body may also have moments about each of the three axes. The second set of three equilibrium equations states that the sum of the moment components about the x, y, and z axes must also be equal to zero.

We break the forces into three component equations

$$\sum \vec{F} = 0 \implies \sum F_x = 0; \sum F_y = 0; \sum F_z = 0$$

We break the moments into three component equations

$$\sum \vec{M} = 0 \implies \sum M_x = 0; \sum M_y = 0; \sum M_z = 0$$

## Finding the Equilibrium Equations:

As with particles, the first step in finding the equilibrium equations is to draw a free body diagram of the body being analyzed. This diagram should show all the force vectors acting on the body. In the free body diagram, provide values for any of the known magnitudes, directions, and points of application for the force vectors and provide variable names for any unknowns (either magnitudes, directions, or distances).

Next you will need to choose the x, y, z axes. These axes do need to be perpendicular to one another, but they do not necessarily have to be horizontal or vertical. If you choose coordinate axes that line up with some of your force vectors you will simplify later analysis.

Once you have chosen axes, you need to break down all of the force vectors into components along the x, y and z directions (see the vectors page in Appendix 1 page for more details on this process). Your first equation will be the sum of the magnitudes of the components in the x direction being equal to zero, the second equation will be the sum of the magnitudes of the components in the y direction being equal to zero, and the third (if you have a 3D problem) will be the sum of the magnitudes in the z direction being equal to zero.

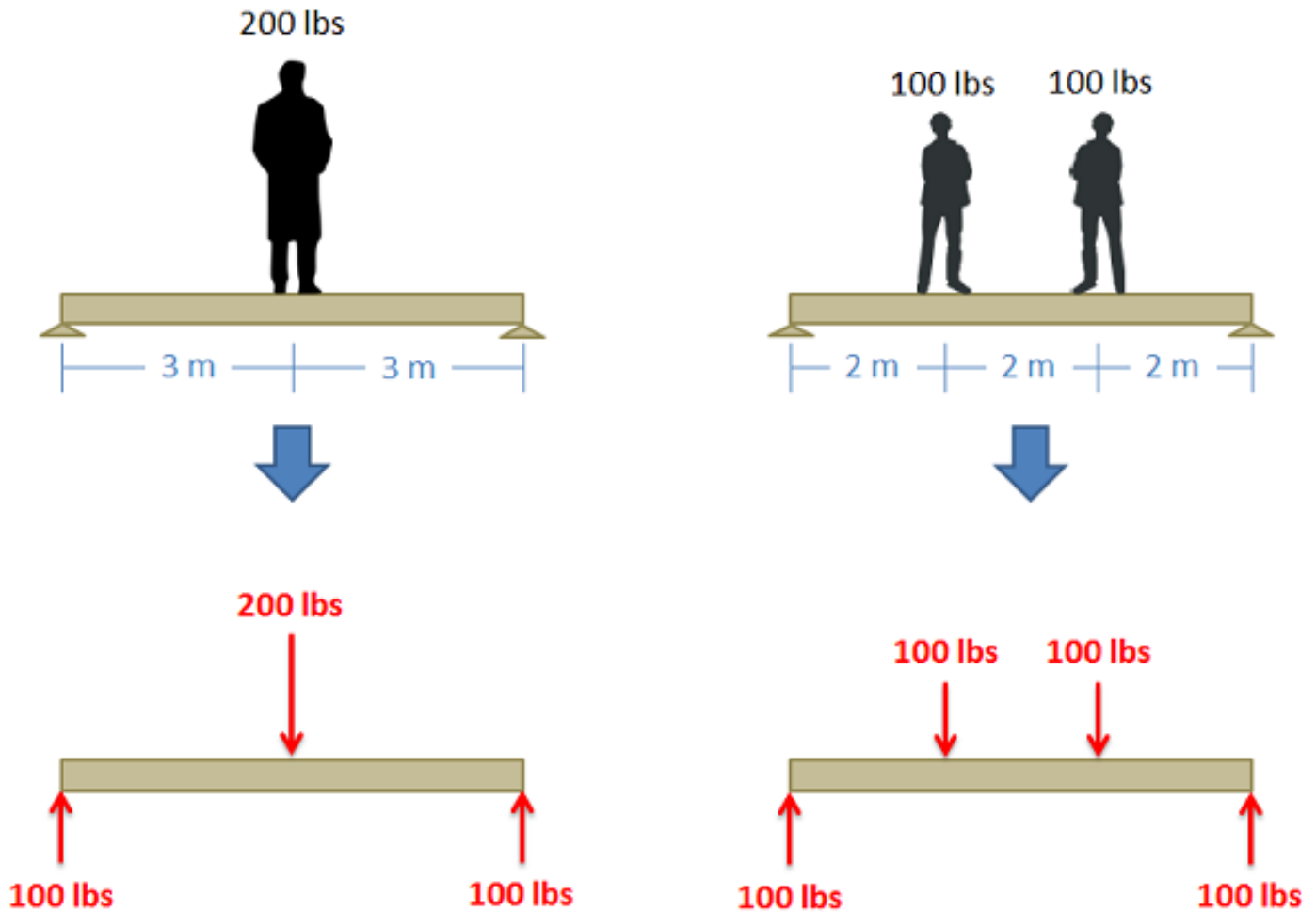
Next you will need to come up with the the moment equations. To do this you will need to choose a point to take the moments about. Any point should work, but it is usually advantageous to choose a point that will decrease the number of unknowns in the equation. Remember that any force vector that travels through a given point will exert no moment about that point. To write out the moment equations simply sum the moments exerted by each force (adding in pure moments shown in the diagram) about the given point and the given axis (x, y, or z) and set that sum equal to zero. All moments will be about the z axis for two dimensional problems, though moments can be about x, y and z axes for three dimensional problems.

Once you have your equilibrium equations, you can solve these formulas for unknowns. The number of unknowns that you will be able to solve for will again be the number or equations that you have.

Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/3\\_equilibrium\\_rigid\\_body/3-6\\_equilibrium\\_equations\\_rigid\\_body/equilibrium\\_equations\\_rigid\\_body.html](http://mechanicsmap.psu.edu/websites/3_equilibrium_rigid_body/3-6_equilibrium_equations_rigid_body/equilibrium_equations_rigid_body.html)

Here is a visual example of using the equilibrium equations:



Source: Engineering Mechanics, Jacob Moore, et al.

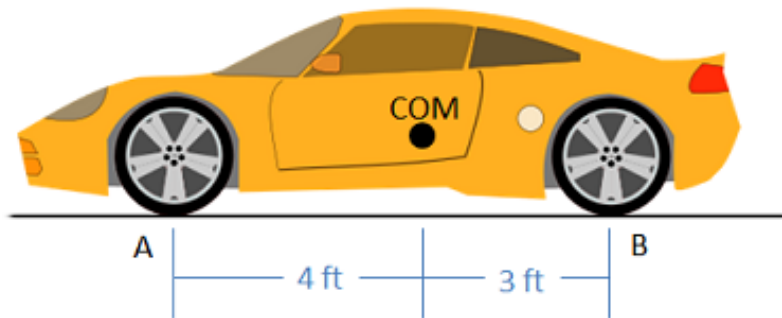
[http://mechanicsmap.psu.edu/websites/4\\_statically\\_equivalent\\_systems/4-1\\_statically\\_equivalent\\_systems/images/equivalentexample.png](http://mechanicsmap.psu.edu/websites/4_statically_equivalent_systems/4-1_statically_equivalent_systems/images/equivalentexample.png)

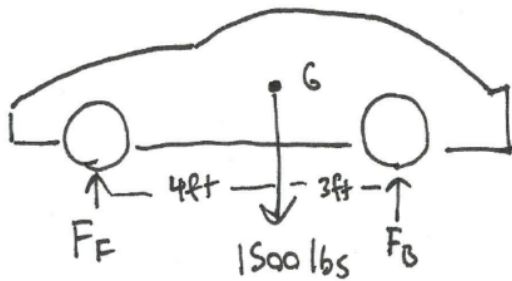
If we only consider the  $y$  (vertical) direction, the 200 lbs pushing down on the beam must be balanced by the reaction forces pushing up. The two reaction forces are equivalent because the forces on top are balanced evenly between the reaction forces. If they are at different locations, we use the sum of the moments equation and the distances of the people to determine the size of the reaction forces.



## Example 1:

The car below has a mass of 1500 lbs with the center of mass 4 ft behind the front wheels of the car. What are the normal forces on the front and the back wheels of the car?





$$\sum F_x = 0 = 0$$

$$\sum F_y = F_F + F_B - 1500 = 0$$

$$\sum M_G = -(4)(F_F) + (3)(F_B) = 0$$

$$F_F = \frac{3}{4} F_B$$

$$\frac{3}{4} F_B + F_B - 1500 = 0$$

$$1.75 F_B = 1500$$

$$F_B = 857.14 \text{ lbs}$$

$$F_F = \frac{3}{4} F_B$$

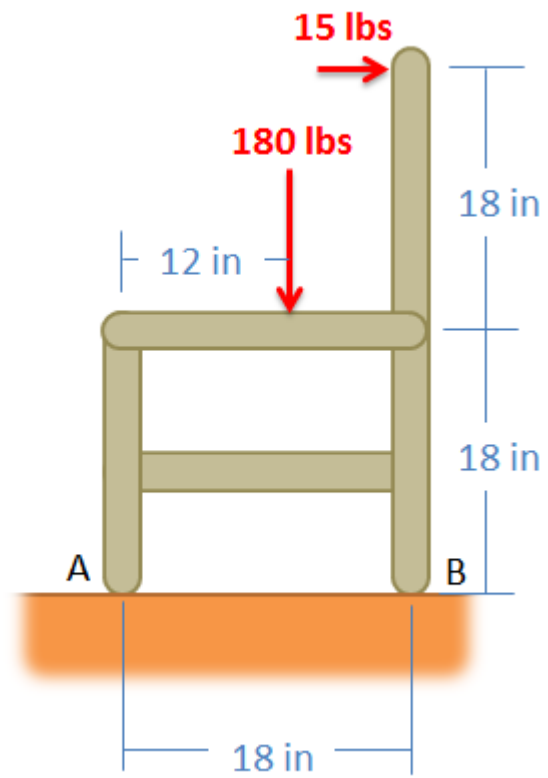
$$F_F = 642.86 \text{ lbs}$$

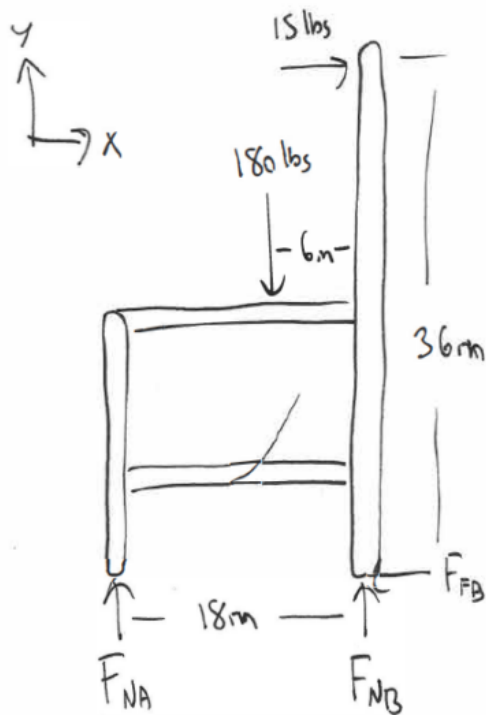
Source: Engineering Mechanics, Jacob Moore et al.,

[http://mechanicsmap.psu.edu/websites/3\\_equilibrium\\_rigid\\_body/3-6\\_equilibrium\\_equations\\_rigid\\_body/pdf/EquilibriumEquationsExtended\\_WorkedProblem1.pdf](http://mechanicsmap.psu.edu/websites/3_equilibrium_rigid_body/3-6_equilibrium_equations_rigid_body/pdf/EquilibriumEquationsExtended_WorkedProblem1.pdf)

## Example 2:

While sitting in a chair, a person exerts the forces in the diagram below. Determine all forces acting on the chair at points A and B. (Assume A is frictionless and B is a rough surface).





$$\sum F_x = 15 - F_{FB} = 0$$

$$\sum F_y = F_{NA} + F_{NB} - 180 = 0$$

$$\sum M_B = -(F_{NA})(18) + (180)(6) - (15)(36) = 0$$

$$F_{NA} = \frac{-(15)(36) + (180)(6)}{18}$$

$$F_{NA} = 30 \text{ lbs}$$

$$F_{FB} = 15 \text{ lbs}$$

$$F_{NB} = 180 - F_{NA}$$

$$F_{NB} = 150 \text{ lbs}$$

Source: Engineering Mechanics, Jacob Moore et al.,  
[http://mechanicsmap.psu.edu/websites/3\\_equilibrium\\_rigid\\_body/3-6\\_equilibrium\\_equations\\_rigid\\_body/pdf/EquilibriumEquationsExtended\\_WorkedProblem5.pdf](http://mechanicsmap.psu.edu/websites/3_equilibrium_rigid_body/3-6_equilibrium_equations_rigid_body/pdf/EquilibriumEquationsExtended_WorkedProblem5.pdf)

### Key Takeaways

*Basically:* The equilibrium equations for rigid bodies are a way to determine unknown forces and moments using known forces and moments, separating the motion in 2 (or 3) directions for translation and rotation. Moments could be calculated because rigid bodies also consider shape and length.

*Application:* Calculate the reaction forces from the combined weight of an object.

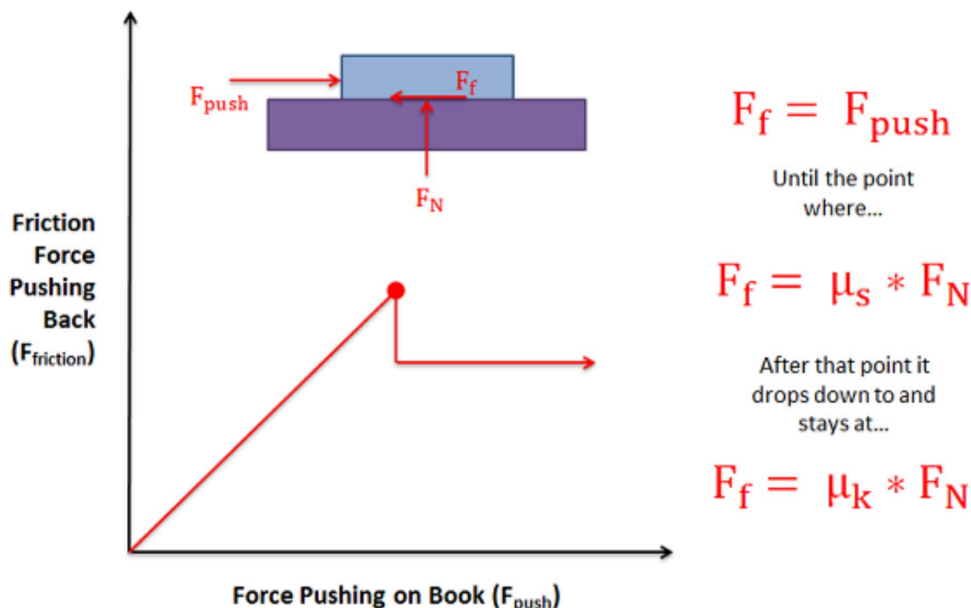
*Looking Ahead:* This method will be used extensively in Ch 5 and 6.

## 4.4 Friction and Impending Motion

### Dry Friction

**Dry friction** is the force that opposes one solid surface sliding across another solid surface. Dry friction always opposes the surfaces sliding relative to one another and can have the effect of either opposing motion or causing motion in bodies.

The most commonly used model for dry friction is **coulomb friction**. This type of friction can further be broken down into static friction and kinetic friction. These two types of friction are illustrated in the diagram below. First imagine a box sitting on a surface. A pushing force is applied parallel to the surface and is constantly being increased. A gravitational force, a normal force, and a frictional force are also acting on the box.



As the pushing force increases, the static friction force will be equal in magnitude and opposite in direction until the point of impending motion. Beyond this point, the box will begin to slip as the pushing force is greater in magnitude than the kinetic friction force opposing

**Static friction** occurs prior to the box slipping and moving. In this region the friction force will be equal in magnitude and opposite in direction to the pushing force itself. As the magnitude of the pushing force increases so does the magnitude of the friction force.

If the magnitude of the pushing force continues to rise, eventually the box will begin to slip. As the box begins to slip the type of friction opposing the motion of the box changes from static friction to what is called kinetic friction. The point just before the box slips is known as **impending motion**. This can also be thought of as the maximum static friction force before slipping. The magnitude of the maximum static friction force is equal to the static coefficient of friction times the normal force existing between the box and the surface. This coefficient of friction is

a property that depends on both materials and can usually be looked up in tables.

**Kinetic friction** occurs beyond the point of impending motion when the box is sliding. With kinetic friction, the magnitude of the friction force opposing motion will be equal to the kinetic coefficient of friction times the normal force between the box and the surface. The kinetic coefficient of friction also depends upon the two materials in contact, but will almost always be less than the static coefficient of friction.

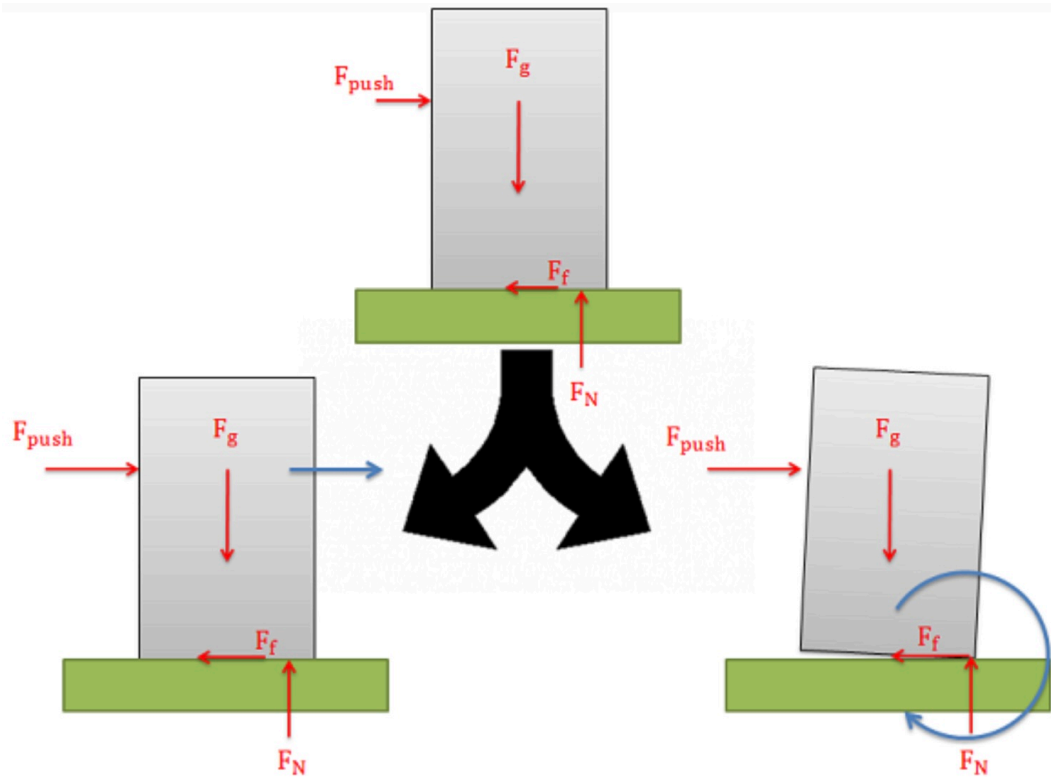
Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/7\\_friction/7-1\\_dry\\_friction/dryfriction.html](http://mechanicsmap.psu.edu/websites/7_friction/7-1_dry_friction/dryfriction.html)

## Slipping vs. Tipping

Imagine a box sitting on a rough surface as shown in the figure below. Now imagine that we start pushing on the side of the box. Initially the friction force will resist the pushing force and box will sit still. As we increase the force pushing the box however, one of two things will occur.

1. The pushing force will exceed the maximum static friction force and the box will begin to slide across the surface (slipping).
2. Or, the pushing force and the friction force will create a strong enough couple that the box will rotate and fall on it's side (tipping).

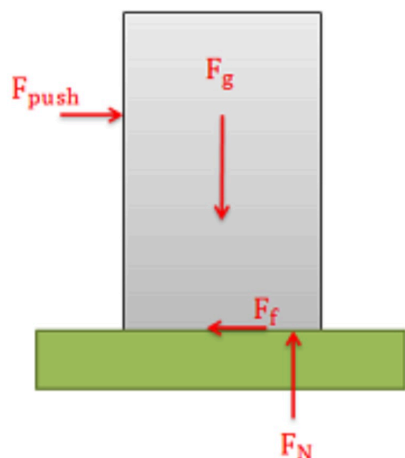


As the pushing force increases on the box, it will either begin to slide along surface (slipping) or it will begin to rotate (tipping)

When we look at cases where either slipping or tipping may occur, we are usually interested in finding which of the two options will occur first. To determine this, we usually determine both the pushing force necessary to make the body slip and the pushing force necessary to make the body tip over. Whichever option requires less force is the option that will occur first.

### Determining the Force Required to Make an Object “Slip”:

A body will slide across a surface if the pushing force exceeds the maximum static friction force that can exist between the two surfaces in contact. As in all dry friction problems, this limit to the friction force is equal to the static coefficient of friction times the normal force between the body. If the pushing force exceeds this value then the body will slip.



$$F_{f(max)} = \mu_s * F_N$$

If

$$F_{push} > F_{f(max)}$$

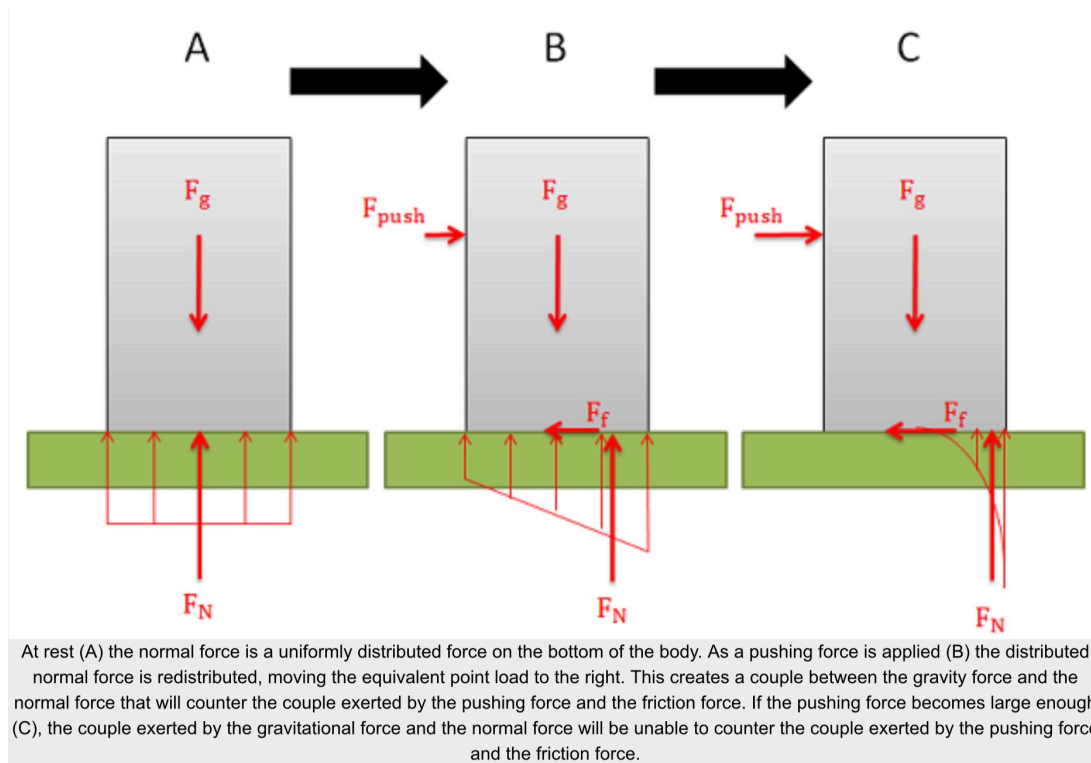
Then the box will  
begin to slide

If the pushing force exceeds the maximum static friction force ( $\mu_s * F_N$ ) then the body will begin to slide.



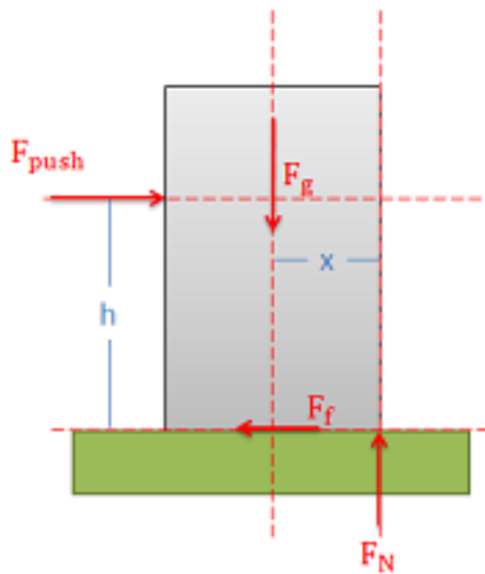
### Determining the Force Required to Make an Object “Tip”:

The normal forces supporting bodies are distributed forces. These forces will not only prevent the body from accelerating into the ground due to gravitational forces, but they can also redistribute themselves to prevent a body from rotating when forces cause a moment to act on the body. This redistribution will result in the equivalent point load for the normal force shifting to one side or the other. A body will tip over when the normal force can no longer redistribute itself any further to resist the moment exerted by other forces (such as the pushing force and the friction force).



The easiest way to think about the shifting normal force and tipping is to imagine the equivalent point load of the distributed normal force. As we push or pull on the body, the normal force will shift to the left or right. This normal force and the gravitational force create a couple that exerts a moment. This moment will be countering the moment exerted by the couple formed by the pushing force and the friction force.

Because the normal force is the direct result of physical contact, we cannot shift the normal force beyond the surfaces in contact (aka. the edge of the box). If countering the moment exerted by the pushing force and the friction force requires shifting the normal force beyond the edge of the box, then the normal force and the gravity force will not be able to counter the moment and as a result the box will begin to rotate (aka. tip over).



$$M_{g/N} = F_g * x$$

$$M_{push/f} = F_{push} * h$$

If

$$M_{push/f} > M_{g/N}$$

Then the box will  
begin to tip over

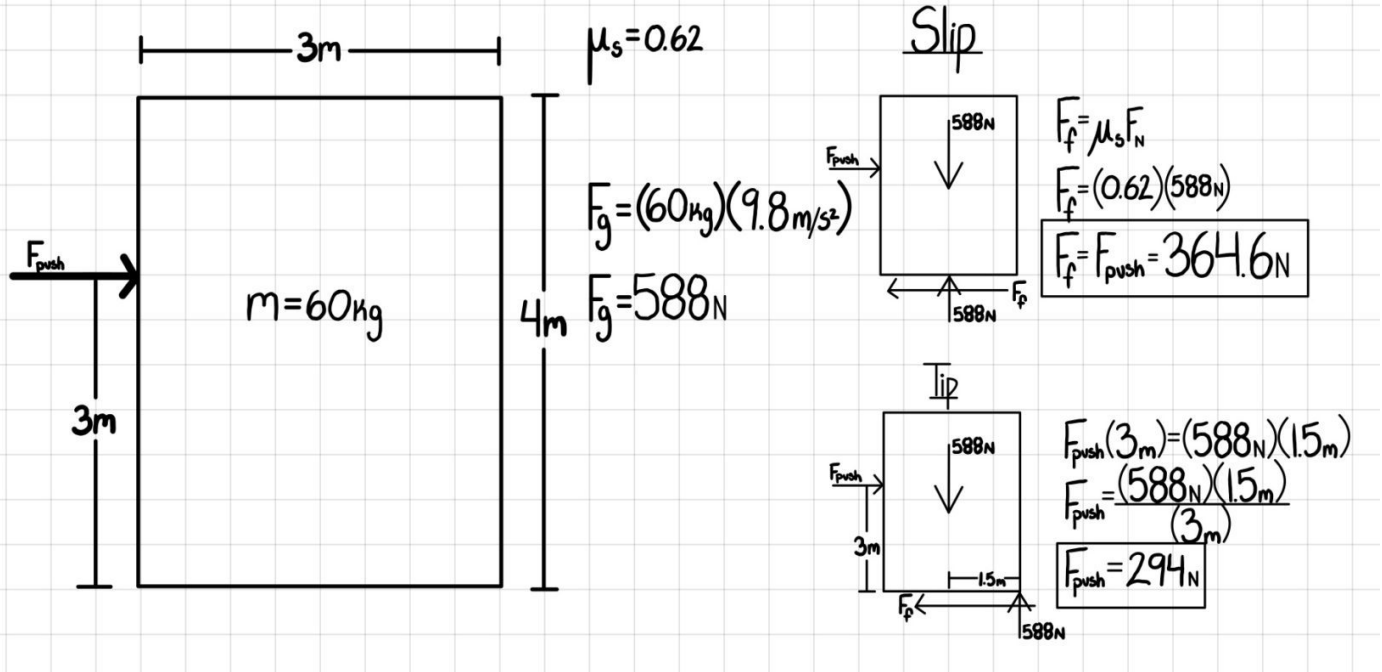
The body will tip when the moment exerted by the pushing and friction forces exceeds the moment exerted by the gravity and normal forces. For impending motion the normal force will be acting at the very edge of the body.

Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/7\\_friction/7-2\\_slipping\\_vs\\_tipping/slippingvstipping.html](http://mechanicsmap.psu.edu/websites/7_friction/7-2_slipping_vs_tipping/slippingvstipping.html)

## Example 1

The box shown below is pushed as shown. If we keep increasing the pushing force, will the box first begin to slide or will it tip over?



Therefore, the box will TIP first.

Source: Gayla Cameron.

## Example 2:

A 500 lb box is sitting on concrete floor. If the static coefficient of friction is .7 and the kinetic coefficient of friction is .6:

- What is the friction force if the pulling force is 150 lbs?
- What pulling force would be required to get the box moving?
- What is the minimum force required to keep the box moving once it has started moving?



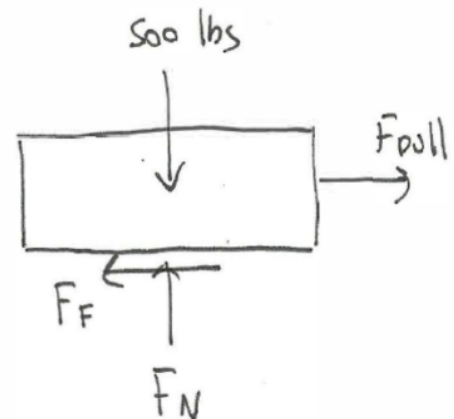
a)

$$F_{F\max} = (\mu_s)(F_N)$$

$$\Sigma F_y = F_N - 500 = 0$$
$$F_N = 500 \text{ lbs}$$

$$F_{F\max} = (.7)(500)$$

$$F_{F\max} = 350 \text{ lbs}$$



If  $F_{\text{pull}} = 150 \text{ lbs}$  then  $F_F = 150 \text{ lbs}$

b) to get it started

$$F_{\text{pull}} = (\mu_s)(F_N) = (.7)(500)$$

$$F_{\text{pull}} = 350 \text{ lbs}$$

c) to keep it going

$$F_{\text{pull}} = (\mu_k)(F_N) = (.6)(500)$$

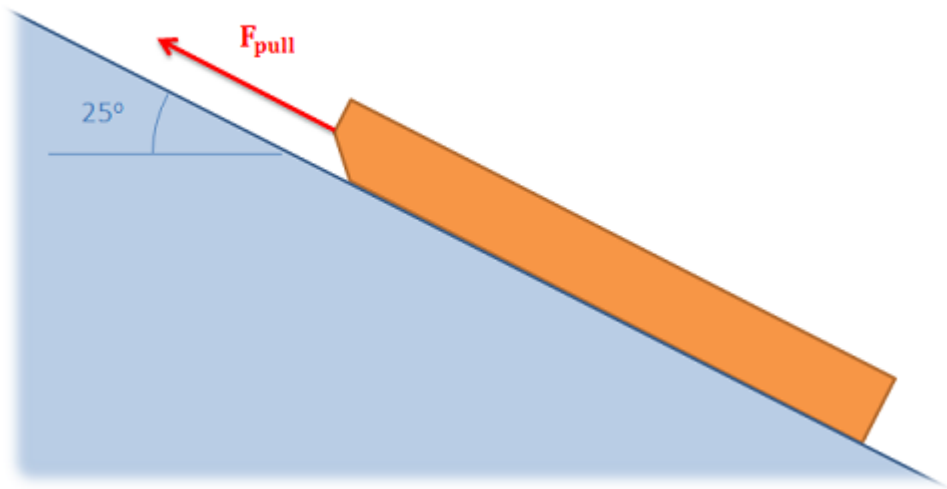
$$F_{\text{pull}} = 300 \text{ lbs}$$

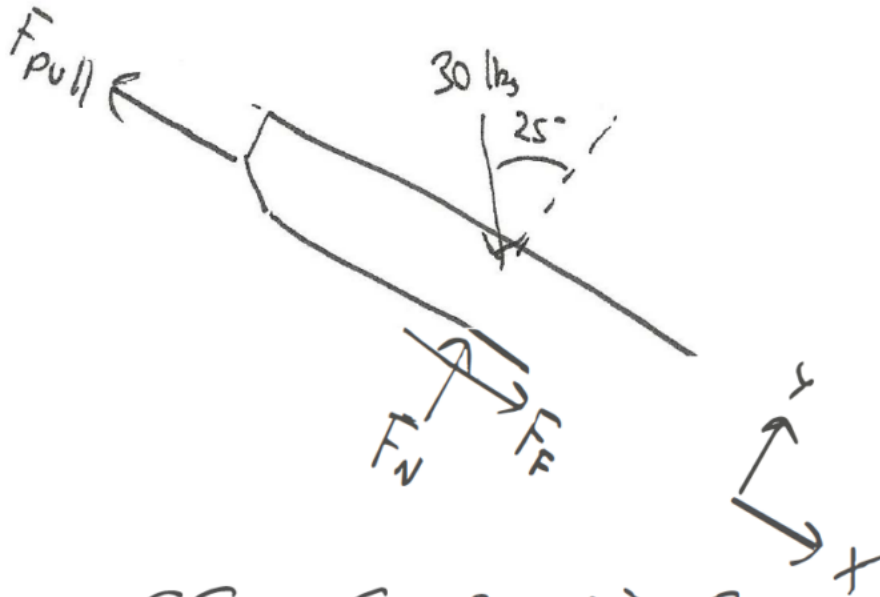
Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/7\\_friction/7-1\\_dry\\_friction/pdf/DryFriction\\_WorkedExample1.pdf](http://mechanicsmap.psu.edu/websites/7_friction/7-1_dry_friction/pdf/DryFriction_WorkedExample1.pdf)

### Example 3:

A 30 lb sled is being pulled up an icy incline of 25 degrees. If the static coefficient of friction between the ice and the sled is .4 and the kinetic coefficient of friction is .3, what is the required pulling force needed to keep the sled moving at a constant rate?





$$\sum F_x = F_F + 30 \sin(25) - F_{\text{pull}} = 0$$

$$\sum F_y = F_N - 30 \cos(25) = 0$$

sliding so  $F_F = (\mu_k)(F_N) = .3 F_N$

$$F_N = 30 \cos(25) = 27.19 \text{ lbs}$$

$$(.3)(27.19) + 30 \sin(25) = F_{\text{pull}}$$

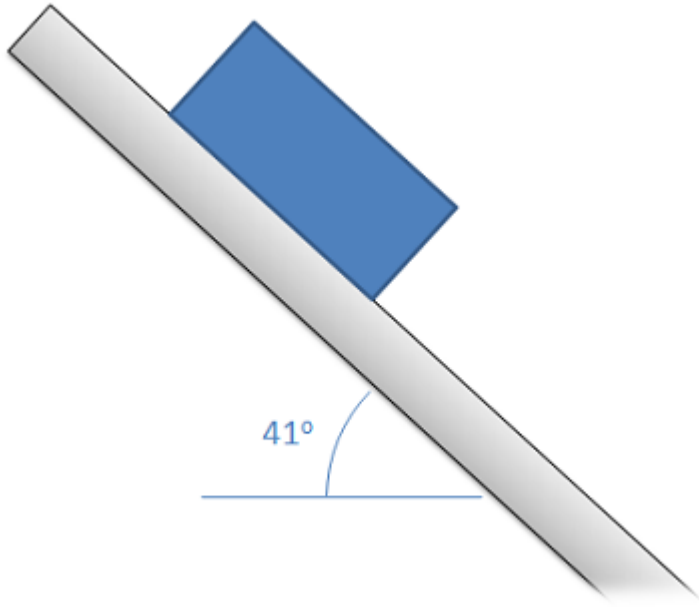
$$F_{\text{pull}} = 20.84 \text{ lbs}$$

Source: Engineering Mechanics, Jacob Moore, et al.

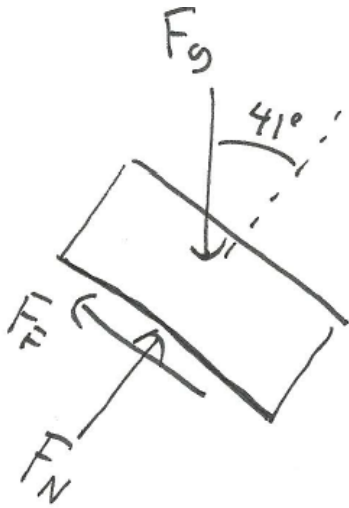
[http://mechanicsmap.psu.edu/websites/7\\_friction/7-1\\_dry\\_friction/pdf/DryFriction\\_WorkedExample2.pdf](http://mechanicsmap.psu.edu/websites/7_friction/7-1_dry_friction/pdf/DryFriction_WorkedExample2.pdf)

## Example 4:

A plastic box is sitting on a steel beam. One end of the steel beam is slowly raised, increasing the angle of the surface until the box begins to slip. If the box begins to slip when the beam is at an angle of  $41^\circ$ , what is the static coefficient of friction between the steel beam and the plastic box?







just before slipping

$$\Sigma F_x = F_g \sin(41) - F_F = 0$$

$$\Sigma F_y = F_N - F_g \cos(41) = 0$$

$$F_F = (\mu_s) F_N$$

$$F_s = \frac{F_F}{\sin(41)} = \frac{(\mu_s)(F_N)}{\sin(41)}$$

$$F_s = \frac{F_N}{\cos(41)}$$

$$\frac{(\mu_s) \cancel{F_N}}{\sin(41)} = \frac{\cancel{F_N}}{\cos(41)}$$

$$\mu_s = \frac{\sin(41)}{\cos(41)} = \tan(41)$$

$$\mu_s = 0.87$$

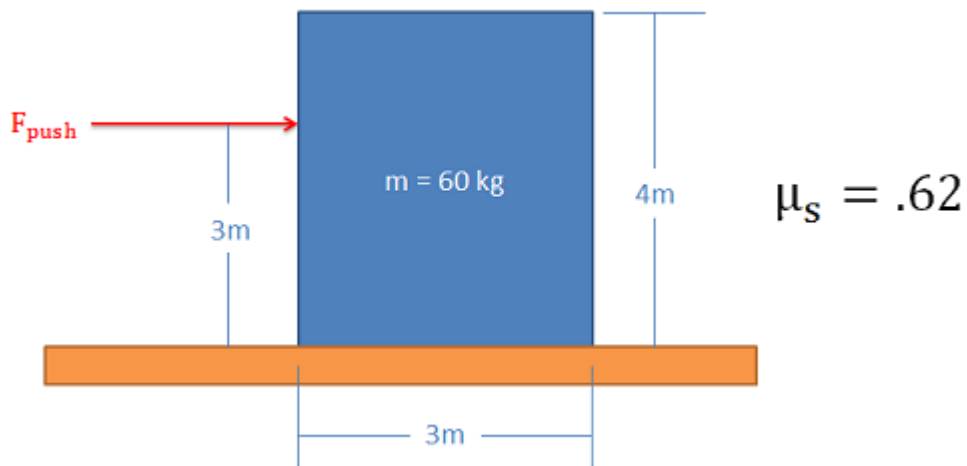
Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/7\\_friction/7-1\\_dry\\_friction/pdf/DryFriction\\_WorkedExample3.pdf](http://mechanicsmap.psu.edu/websites/7_friction/7-1_dry_friction/pdf/DryFriction_WorkedExample3.pdf)

## Example 5: Slipping vs Tipping

Explanation: If it's tipping, all of the normal force will be at the corner. If it starts slipping, it must overcome the static frictional force. Comparing the pushing force needed to tip or slip, the pushing force is lower to cause tipping occurs than the pushing force to cause slipping, there fore it will tip first.

The box shown below is pushed as shown. If we keep increasing the pushing force, will the box first begin to slide or will it tip over?



Slip

$F_g = (60)(9.8)$   
 $F_g = 588 \text{ N}$

Tip

$F_g = 588$

$F_F = \mu_s F_N$   
 $F_F = (.62)(588)$   
 $F_F = F_{\text{push}} = 364.6 \text{ N}$

$(F_{\text{push}})(3) = (588)(1.5)$   
 $F_{\text{push}} = \frac{(588)(1.5)}{3}$   
 $F_{\text{push}} = 294 \text{ N}$

The box will tip first

Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/7\\_friction/7-2\\_slipping\\_vs\\_tipping/pdf/TippingVsSlipping\\_WorkedExample1.pdf](http://mechanicsmap.psu.edu/websites/7_friction/7-2_slipping_vs_tipping/pdf/TippingVsSlipping_WorkedExample1.pdf)

### Key Takeaways

*Basically:* Friction always opposes motion. The coefficient of static friction is always higher than the coefficient of kinetic friction.

*Application:* Slipping and tipping are interesting cases looking at friction. Depending on the mass, the height of the applied force, and the frictional surface, you can calculate whether the object will tip or slip first.

*Looking Ahead:* This will become important in Dynamics.

## 4.5 Examples

Here are examples from Chapter 4 to help you understand these concepts better. These were taken from the real world and supplied by FSDE students in Summer 2021. If you'd like to submit your own examples, please send them to the author [eosgood@upei.ca](mailto:eosgood@upei.ca).

### Example 4.5.1: External Forces, submitted by Elliott Fraser

#### 1. Problem

Billy (160 lbs), Bobby (180 lbs), and Joe (145 lbs) are walking across a small bridge with a length of 11 feet. Both sides of the bridge are supported by rollers. Billy is 2 feet along the bridge whereas Joe is 9 feet along the bridge. If the maximum force that the left side of the bridge can withstand without failing is 225 lbs, where along the bridge can Bobby stand?

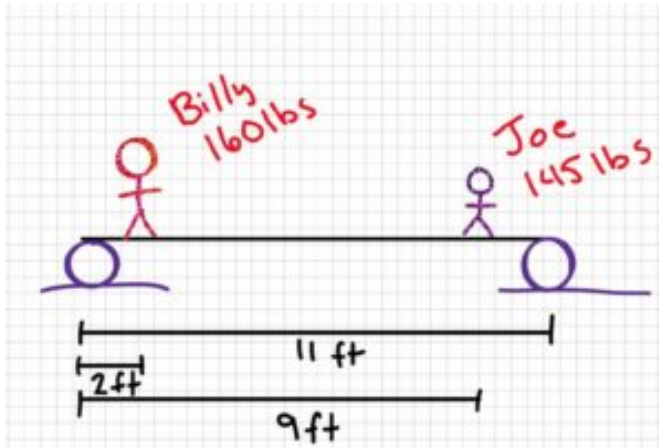
Real-life scenario:



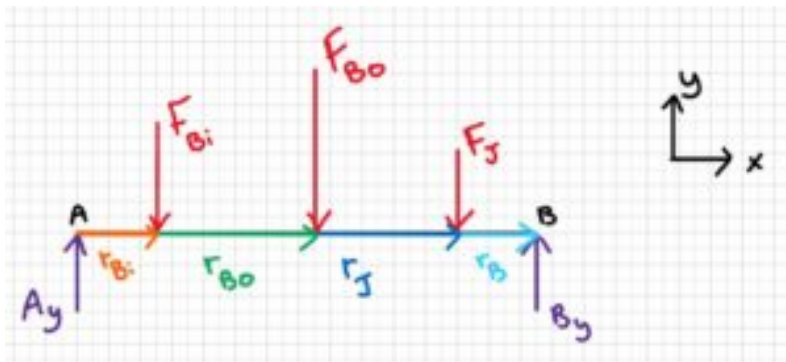
Source: <https://www.flickr.com/photos/chumlee/48306801162/>

**2. Draw**

Sketch:



Free-body diagram:

**3. Knowns and Unknowns**

Known:

- $r_{Bi} = 2 \text{ ft}$
- $r_J = 9 \text{ ft}$
- $r_B = 11 \text{ ft}$
- $F_{Bi} = 160 \text{ lb}$
- $F_J = 145 \text{ lb}$
- $F_{Bo} = 180 \text{ lb}$
- $A_y = 225 \text{ lb}$  (since this is the maximum force without failure)

Note: Since the mass of the bridge was not given, we assume it is negligible and ignore it for this question.

Unknown:  $r_{Bo}$ **4. Approach**

Use equilibrium equations ( $\sum \underline{F} = 0$ ), ( $\sum \underline{M} = 0$ ). Use sum of forces in y to find  $B_y$ ; use sum of moments to find where Bobby can stand. Solve for x.

**5. Analysis**

$$\sum F_y = 0 = -F_{Bi} - F_{Bo} - F_J + A_y + B_y \quad B_y = F_{Bi} + F_{Bo} + F_J - A_y \quad B_y = 160 \text{ lb} + 180 \text{ lb} + 145 \text{ lb} - 225 \text{ lb} \quad B_y = 260 \text{ lb}$$

$$\sum M_A = 0 = -(F_{Bi})(r_{Bi}) - (F_{Bo})(r_{Bo}) - (F_J)(r_J) + (B_y)(r_B) \quad r_{Bo} = \frac{-(F_{Bi})(r_{Bi}) - (F_J)(r_J) + (B_y)(r_B)}{F_{Bo}} = \frac{-(160 \text{ lb})(2 \text{ ft}) - (145 \text{ lb})(9 \text{ ft}) + (260 \text{ lb})(11 \text{ ft})}{180 \text{ lb}}$$

$$\underline{r_{Bo} = 6.86 \text{ ft}}$$

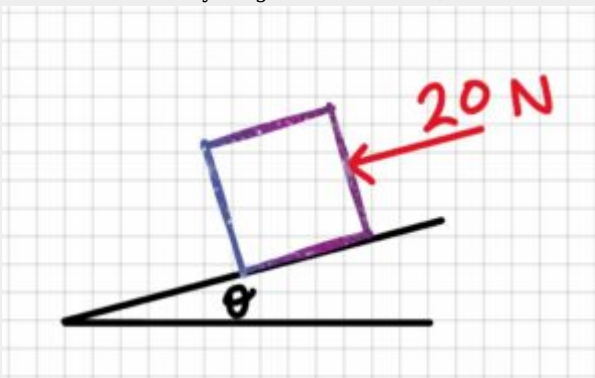
### 6. Review

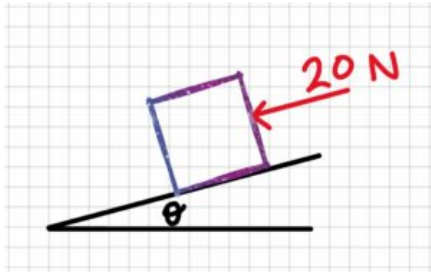
Bobby can stand anywhere from 6.8611 ft – 11 ft from A with no problems. If Bobby were to stand between 0 ft and 6.811 ft, the left side of the bridge would fail.

## Example 4.5.2: Free-Body Diagrams, submitted by Victoria Keefe

### 1. Problem

A box is sitting on an inclined plane ( $\theta = 15^\circ$ ) and is being pushed down the plane with a force of 20 N. Draw the free body diagram for the box, while it is in static equilibrium.



**2. Draw****3. Knowns and Unknowns**

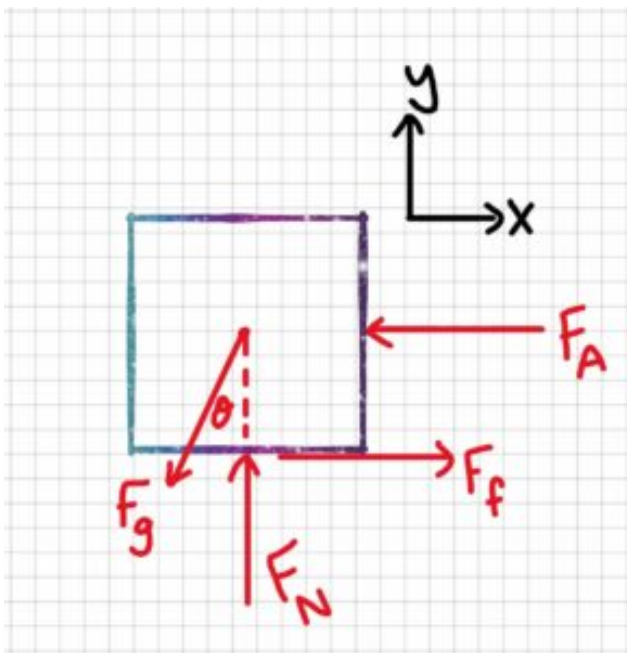
$$\theta = 15^\circ$$

$$F_A$$

Unknowns: free-body diagram of box

**4. Approach**

Draw the box, then draw all forces acting on it

**5. Analysis****6. Review**

All forces acting upon the box are drawn, including weight/gravitational force, normal force, friction, and applied forces.



## Example 4.5.3: Friction, submitted by Deanna Malone

### 1. Problem



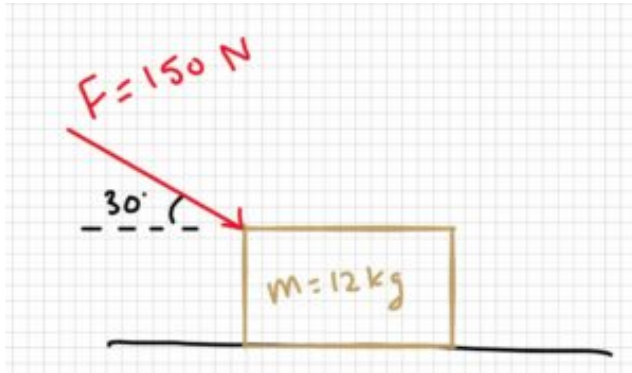
Source:  
<https://static.thenounproject.com/png/2745226-200.png>

A box is being pushed along level ground with a force of 150 N at an angle of  $30^\circ$  with the horizontal. The mass of the box is 12 kg.

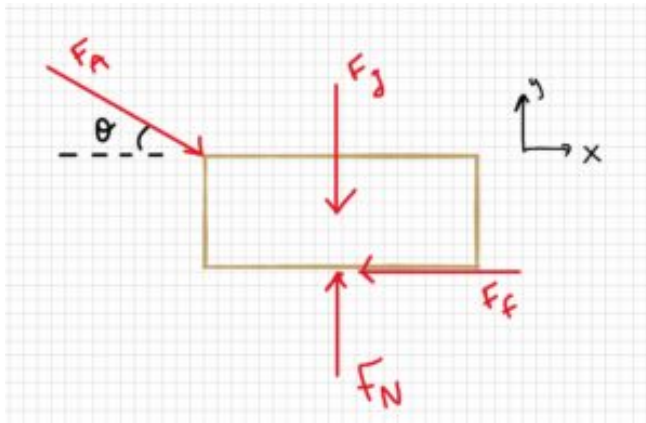
- a) What is the normal force between the box and the floor?
- b) What is the coefficient of friction between the box and the floor?

### 2. Draw

Sketch:



Free Body Diagram:



### 3. Knowns and Unknowns

Knowns:

- $F_A = 150 \text{ N}$
- $\theta = 30^\circ$
- $m = 12 \text{ kg}$

Unknowns:  $F_N, \mu$

### 4. Approach

Use equilibrium equations (  $\sum \underline{F} = 0$  ,  $\sum \underline{M} = 0$  ), SOH CAH TOA, friction equation

### 5. Analysis

Part a:

Find  $F_g$ :

$$F_g = m \cdot g = (12 \text{ kg})(9.81 \text{ m/s}^2) \\ F_g = 117.72 \text{ N}$$

Find  $F_N$  using equilibrium equations:

$$\sum F_y = 0 = F_N - F_g - F_A \sin 30^\circ \\ 0 = F_N - 117.72 \text{ N} - 150 \text{ N} \cdot \sin 30^\circ \\ F_N = 117.72 + 150 \text{ N} \cdot \sin 30^\circ \\ \underline{F_N = 192.7 \text{ N}}$$

Part b:

Find two equations for  $F_f$ , set equal, solve for  $\mu$

$$\sum F_x = 0 = F_A \cos 30^\circ - F_f \quad 0 = 150\text{N} \cdot \cos 30^\circ - F_f$$

$$F_f = 150\text{N} \cdot \cos 30^\circ \quad F_f = \mu \cdot F_N \quad 150\text{N} \cdot \cos 30^\circ = \mu \cdot F_N$$

$$\mu = \frac{150\text{N} \cdot \cos 30^\circ}{F_N} \quad \mu = \frac{150\text{N} \cdot \cos 30^\circ}{192.72\text{N}} = 0.67405$$

$$\underline{\mu = 0.67}$$

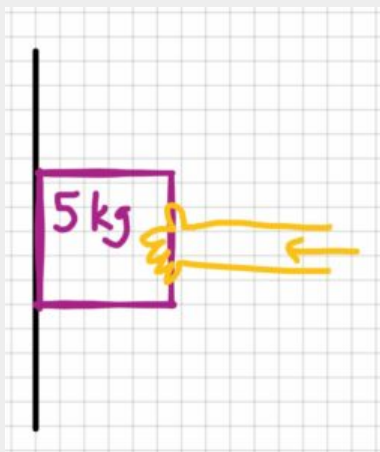
## 6. Review

$F_N$ ,  $F_g$ , and the y component of  $F_A$  are the only forces in the y direction so it makes sense that they need to equal zero for the equilibrium equations.  $F_N$  is the only positive force in the y direction, so it makes sense that it equals the magnitude of the other two put together.

The coefficient found between the box and the floor is reasonable as it is less than 1, and it's reasonable for a box on the floor. For example, if the box was wood and the floor was wood, the coefficient of static friction would be anywhere from 0.5-0.7, so having a coefficient of friction being equal to 0.67 makes sense.

# Example 4.5.4: Friction, submitted by Dhruvil Kanani

## 1. Problem



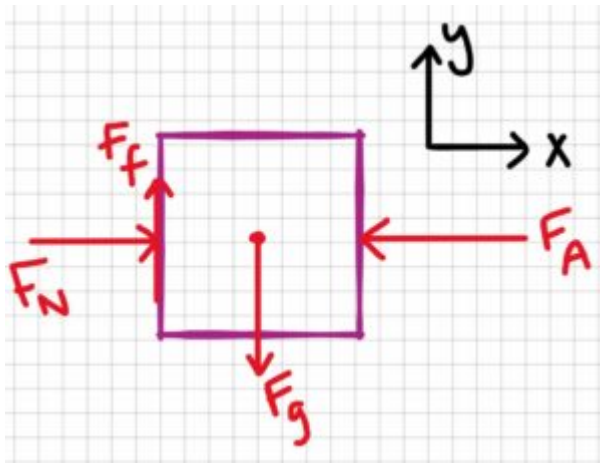
A person is trying to prevent a brick from sliding on a rough vertical surface by applying force in the direction of wall. Assuming the coefficient of static friction is 0.49 and mass of the brick is 5 kg,

- a) Determine the minimum force required to prevent the brick from slipping.
- b) Find the distributed load or intensity if the length of the person's hands from the tip of his fingers to their wrist is 16 cm.

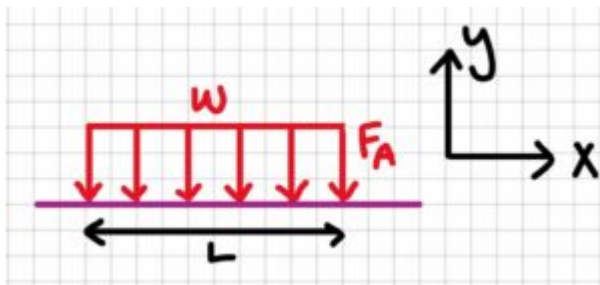
## 2. Draw

Sketch:

Free-body diagram (box):



Free-body diagram (distributed load):



## 3. Knowns and Unknowns

Knowns:

- Mass of brick ( $m$ ) = 5 kg
- Coefficient of friction ( $\mu_s$ ) = 0.49
- Acceleration due to gravity ( $g$ ) = 9.8 m/s<sup>2</sup>
- Length of the hand ( $L$ ) = 16 cm

Unknowns:

Applied force ( $F_A$ ), intensity ( $w$ )

#### 4. Approach

Use equilibrium equations (  $\sum \underline{F} = 0$  ,  $\sum \underline{M} = 0$  ), equations for  $F_g$  and  $F_f$  (see below).

$$F_g = m \cdot g \quad F_f = \mu F_N$$

#### 5. Analysis

Part a:

$$F_g = m \cdot g \quad F_g = 5 \text{ kg} \cdot 9.81 \text{ m/s}^2 \quad F_g = 49.05 \text{ N}$$

$$\sum F_y = 0 = -F_g + F_f \quad F_f = F_g \quad F_f = 49.05 \text{ N}$$

$$F_f = \mu_1 F_N \quad F_N = \frac{F_f}{\mu_1} \quad F_N = \frac{49.05 \text{ N}}{0.49}$$

$$\underline{F_N = 100.1 \text{ N}}$$

Part b:

$$\sum F_x = 0 = F_N - F_A \quad F_A = F_N \quad F_A = 100.1 \text{ N}$$

$$w = \frac{F}{L} \quad w = \frac{100.1 \text{ N}}{(16 \text{ cm}) \cdot \frac{1 \text{ m}}{100 \text{ cm}}}$$

$$\underline{w = 625 \text{ N/m}}$$

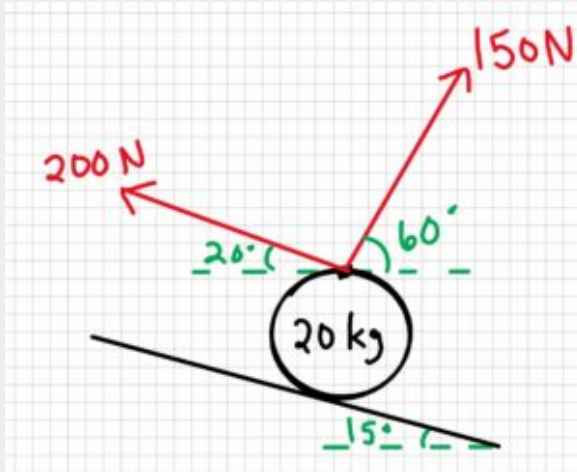
#### 6. Review

It makes sense that the applied force is larger than the gravitational force. It also makes sense that the normal and applied forces are equal, since they are the only forces in the x direction (same goes for the friction and gravitational forces).

## Example 4.5.5: Friction, submitted by Emma

# Christensen

## 1. Problem

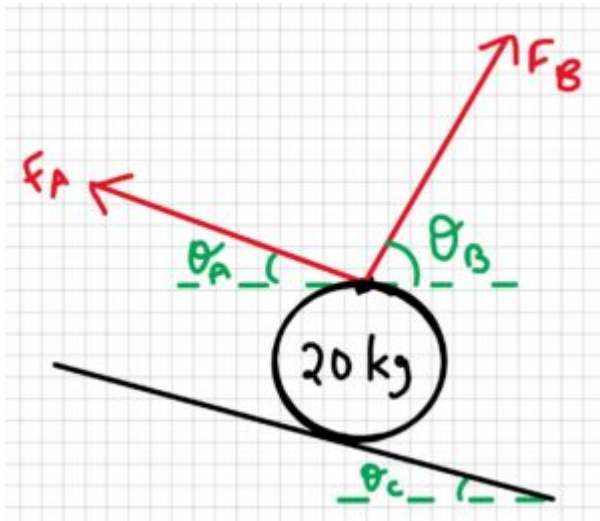


A ball is suspended by two ropes, and rests on an inclined surface with angle  $15^\circ$ . Rope A pulls on the ball with force 200 N, and rope B has force 150 N. They each have angles of  $20^\circ$  and  $60^\circ$  from the inclined surface plane, as shown in the image below.

- Draw a free-body diagram of the ball
- Find the friction force

## 2. Draw

Sketch:



### 3. Knowns and Unknowns

- $\theta_A = 20^\circ$
- $\theta_B = 60^\circ$
- $\theta_C = 15^\circ$
- $m = 20 \text{ kg}$
- $F_A = 200 \text{ N}$
- $F_B = 150 \text{ N}$

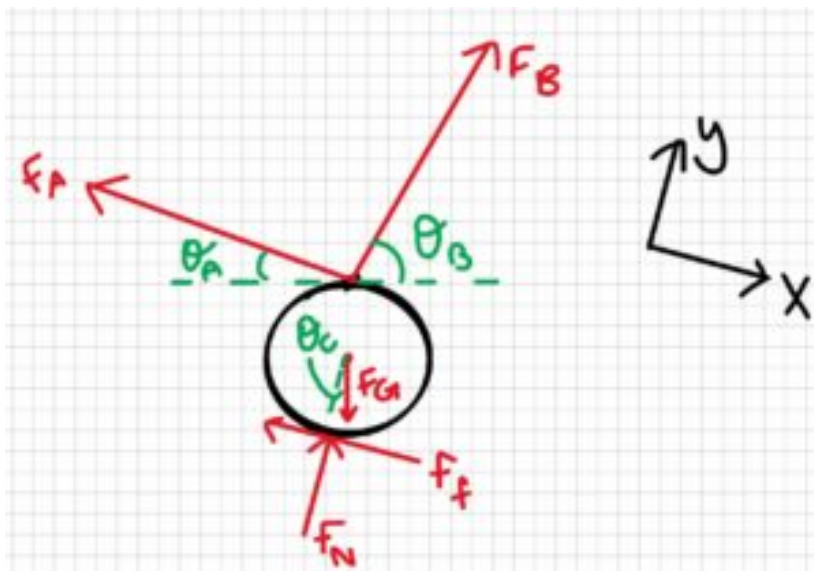
Unknowns:  $F_f$

### 4. Approach

Draw the ball, then add forces. Use equilibrium equations (  $\sum \underline{F} = 0$  ,  $\sum \underline{M} = 0$  ) to find the friction force.

### 5. Analysis

Part a:

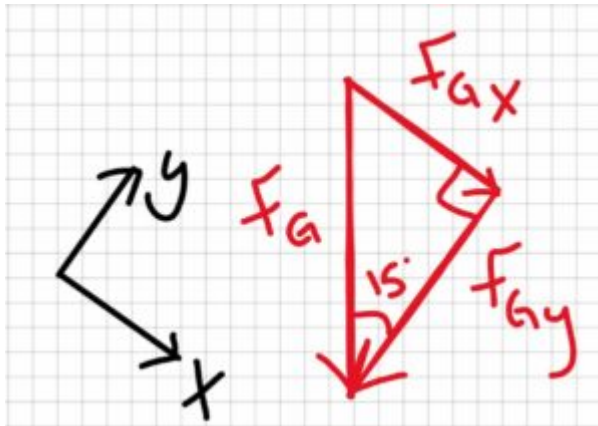


Part b:

Step 1: find  $F_G$

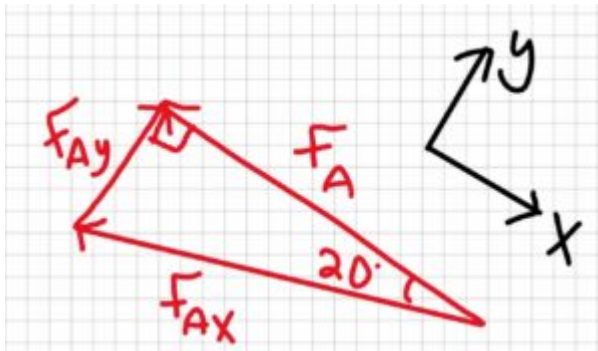
$$F_G = m \cdot g = 20 \text{ kg} \cdot 9.81 \text{ m/s}^2 = 196.2 \text{ N}$$

Step 2: Find the x-component of  $F_G$



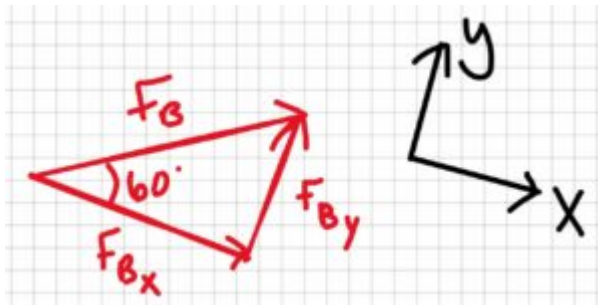
$$F_{GX} = F_G \sin(15^\circ) = 196.2 \text{ N} \cdot \sin(15^\circ) = 50.78 \text{ N}$$

Step 3: Find the x-component of  $F_A$



$$\cos(20^\circ) = \frac{F_{AX}}{F_A} \Rightarrow F_{AX} = \frac{F_A}{\cos(20^\circ)} = \frac{200 \text{ N}}{\cos(20^\circ)} = 212.8 \text{ N}$$

Step 4: Find the x-component of  $F_B$



$$F_{BX} = F_B \cos(60^\circ) = 150 \text{ N} \cdot \cos(60^\circ) = 75 \text{ N}$$

Step 5: Sum forces in the x-direction to find the frictional force

$$\sum F_x = 0 = -f + F_{BX} - F_{AX} + F_{GX} \Rightarrow f = -F_{AX} + F_{BX} + F_{GX} = -212.8 \text{ N} + 75 \text{ N} + 50.78 \text{ N}$$



$$\underline{F_f = -87.02\text{N}}$$

Because the frictional force is negative, that means the frictional force actually acts in the opposite direction, so the friction is keeping the ball from going up the plane.

## 6. Review

The units of  $F_i$  are newtons, which makes sense because it is a force. It also makes sense that  $F_{Ax}$  is larger than  $F_{Gx}$  and  $F_{Bx}$ .

# CHAPTER 5: TRUSSES

This chapter will introduce you to a special type of structure called a 'truss'. You'll analyze these structures more in your Structures course, but for Statics you will need to know how to calculate the force in each member, using two methods: method of joints and method of sections. At first this might seem confusing, but there is something quite elegant and magical about the method once you understand it. Here are the sections in this Chapter:

- [5.1 Trusses Introduction](#) - what are trusses?
- [5.2 Method of Joints](#) - one method of finding the forces in the truss
- [5.3 Method of Sections](#) - another method to find the forces in the truss
- [5.4 Zero-Force Members](#) - how to identify members with no forces
- [5.5 Examples](#) - Examples from your peers

Here are the important equations for this chapter:

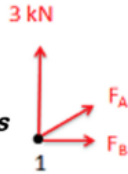
## Ch 5 Concepts & Equations

$$\sum F_x = 0 \quad \sum F_y = 0, \quad \sum M = 0$$

### Method of Joints

#### 1) Solve for reactions using system Eq Eq's

- Draw system FBD
- Sum forces in  $X, Y$
- Sum moments about point with most unknown reaction forces



#### 2) Solve for each joint summing forces in only x and y (no moment)

- Draw a FBD at each joint
- Sum forces in  $X$  and  $Y$  at each joint

- Pick a joint where you have at least 1 known and no more than 2 unknown forces
- If you assume all forces are in tension, then if it comes out negative, it's in compression



### Method of Sections

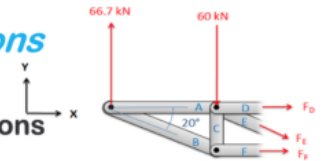
#### 1) Solve for reactions using equilibrium equations

- Draw system FBD
- Sum forces in  $X, Y$
- Sum moments about point with most unknown reaction forces

#### 2) Make a cut & solve for unknowns using equilibrium equations (including moments)

- Make cut & pick a section ( $L$  or  $R$ )
- Draw a FBD of section
- Sum forces in  $X, Y$  and moments of section

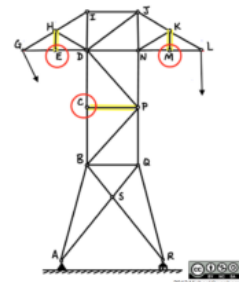
- Use when you only need to know a few members
- Don't always need reaction forces



### Zero Force Members

Carry no load,  
Only for stability

$$\sum F_x = 0 \quad \sum F_y = 0$$



# 5.1 Trusses Introduction

Trusses are rigid structures made up of two-force members, which are objects with exactly two forces/connections. Trusses are commonly found in the frame of a roof and the sides of a bridge:



Source: *Engineering Mechanics, Jacob Moore, et al.*  
[http://mechanicsmap.psu.edu/websites/5\\_structures/5-1\\_structures/structures.html](http://mechanicsmap.psu.edu/websites/5_structures/5-1_structures/structures.html)

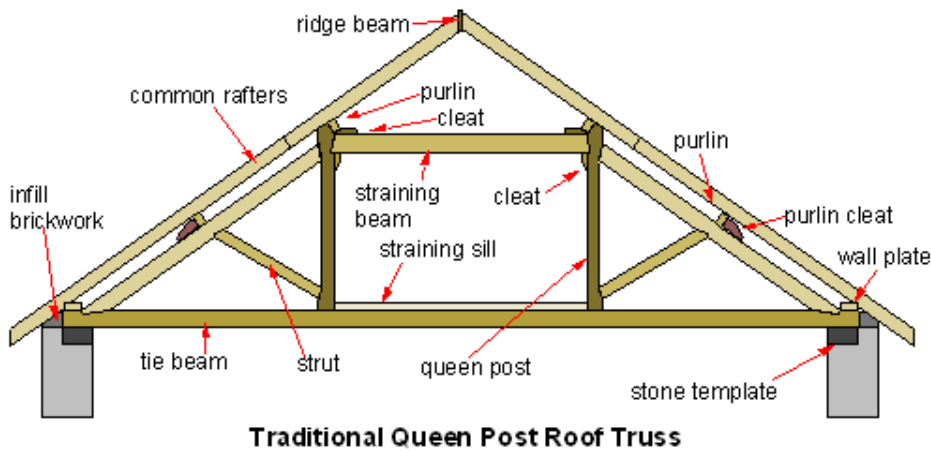


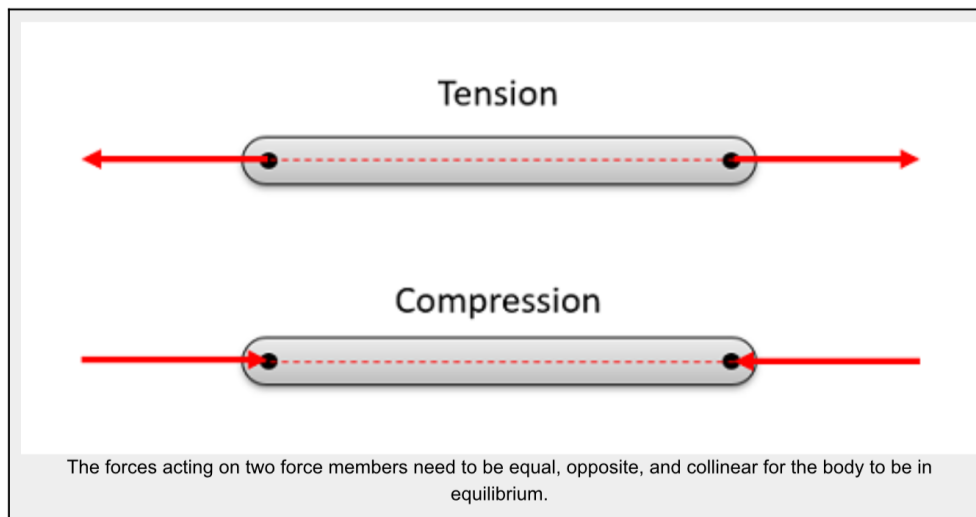
Image Source: *Billbeee at English Wikipedia.* - Transferred from *en.wikipedia* to Commons., CC BY-SA 3.0,

You'll analyze these structures more in your Structures course, but for Statics you will need to know how to calculate the force in each member, using two methods: method of joints and method of sections. Method of joints is more like a particle analysis wherein you use only x and y equilibrium equations. Method of sections is more like a rigid body analysis where you can also include the moment equilibrium equation. Those are in the next sections.

## 5.1.1 Two Force Members

Before we discuss the structure of trusses, we must begin with defining two force members:

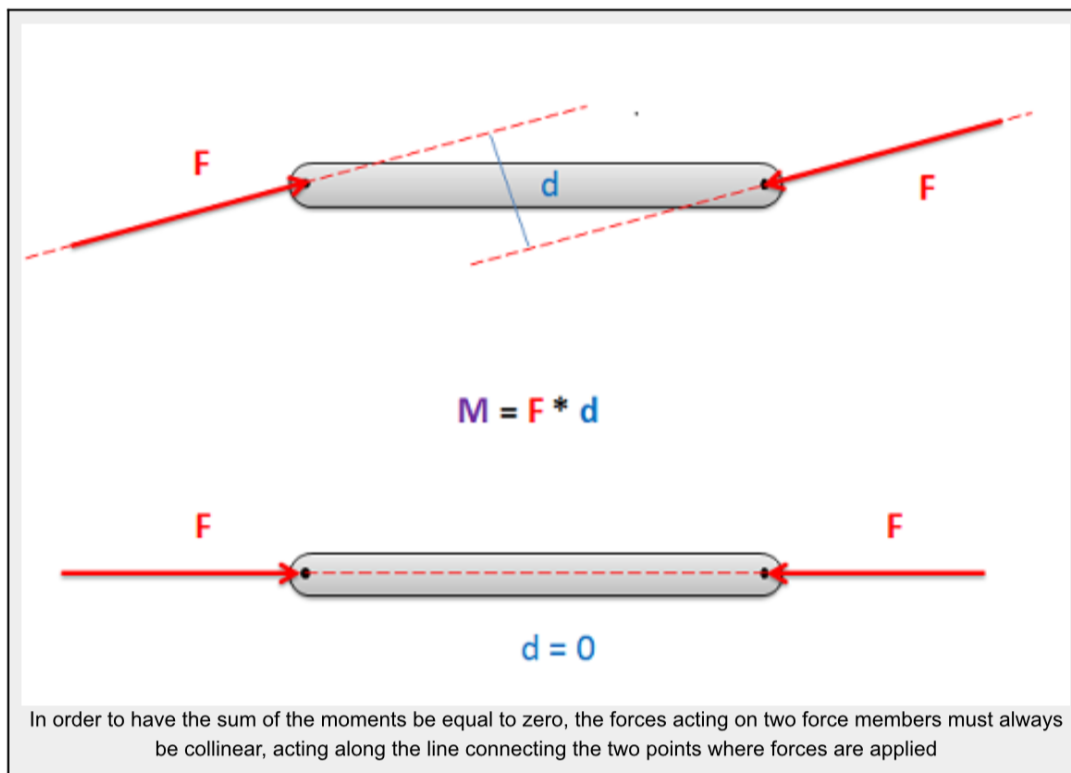
A **two force member** is a body that has forces (and only forces, no moments) acting on it in only two locations. In order to have a two force member in static equilibrium, the net force at each location must be equal, opposite, and collinear. This will result in all two force members being in either tension or compression as shown in the diagram below.



Imagine a beam where forces are only exerted at each end of the beam (a two force member). The body has some non-zero force acting at one end of the beam, which we can draw as a force vector. If this body is in equilibrium, then we know two things: 1) the sum of the forces must be equal to zero, and 2) the sum of the moments must be equal to zero.

In order to have the sum of the forces equal to zero, the force vector on the other side of the beam must be equal in magnitude and opposite in direction. This is the only way to ensure that the sum of the forces is equal to zero with only two forces.

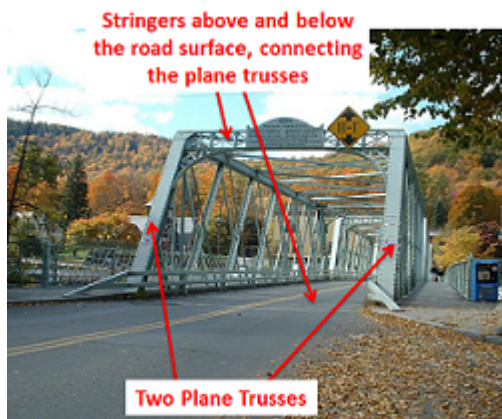
In order to have the sum of the moments equal to zero, the forces must be collinear. If the forces were not collinear, then the two equal and opposite forces would form a couple. This couple would exert a moment on the beam when there are no other moments to counteract the couple. Because the moment exerted by the two forces must be equal to zero, the perpendicular distance between the forces ( $d$ ) must be equal to zero. The only way to achieve this is to have the forces be collinear.



Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/5\\_structures/5-1\\_structures/structures.html](http://mechanicsmap.psu.edu/websites/5_structures/5-1_structures/structures.html)

## 5.1.2 Trusses



Adapted from image by ToddC4176 CC-BY-SA 3.0

A **truss** is an engineering structure that is made entirely of **two force members**. In addition, statically determinate trusses (trusses that can be analyzed completely using the equilibrium equations), must

be **independently rigid**. This means that if the truss was separated from its connection points, no one part would be able to move independently with respect to the rest of the truss.

When we talk about analyzing a truss, we are usually looking to identify not only the external forces acting on the truss structure, but also the forces acting on each member internally in the truss. Because each member of the truss is a two force member, we simply need to identify the magnitude of the force on each member, and determine if each member is in tension or compression.

To determine these unknowns, we have two methods available: the **method of joints**, and the **method of sections**. Both will give the same results, but each through a different process.

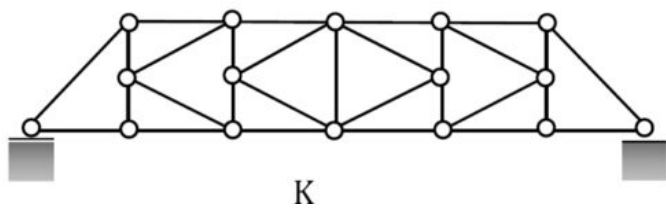
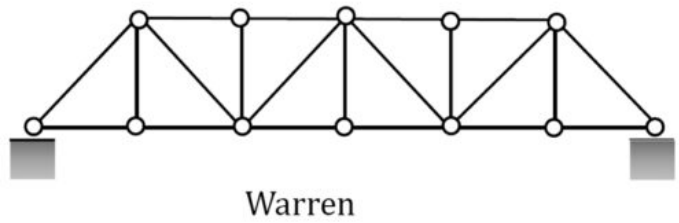
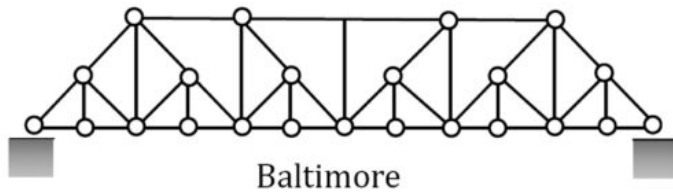
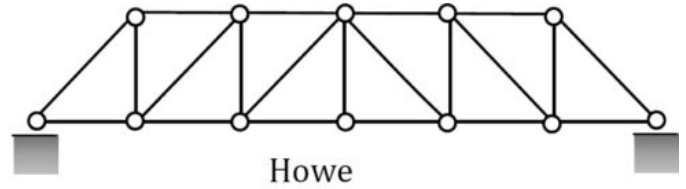
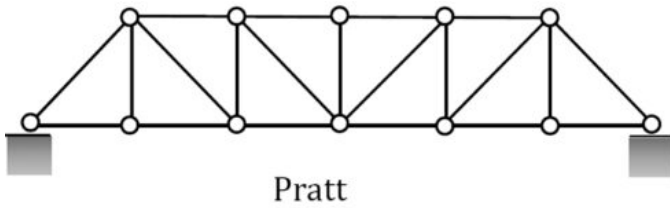
The method of joints focuses on the joints, or the connection points where the members come together. We assume we have a pin at each of these points that we model as a particle, we draw out the free body diagram for each pin, and then write out the equilibrium equations for each pin. This will result in a large number of equilibrium equations that we can use to solve for a large number of unknown forces.

The method of sections involves pretending to split the truss into two or more different sections and then analyzing each section as a separate rigid body in equilibrium. In this method we determine the appropriate sections, draw free body diagrams for each section, and then write out the equilibrium equations for each section.

The method of joints is usually the easiest and fastest method for solving for all the unknown forces in a truss. The method of sections on the other hand is better suited to targeting and solving for the forces in just a few members without having to solve for all the unknowns. In addition, these methods can be combined if needed to best suit the goals of the problem solver.

Source: Engineering Mechanics, Jacob Moore, et al.  
[http://mechanicsmap.psu.edu/websites/5\\_structures/5-3\\_trusses/trusses.html](http://mechanicsmap.psu.edu/websites/5_structures/5-3_trusses/trusses.html)

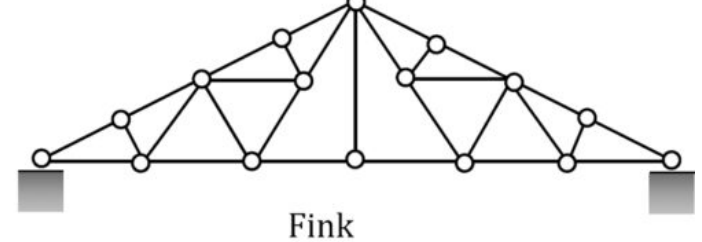
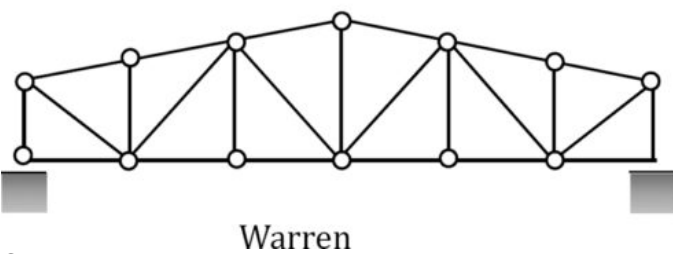
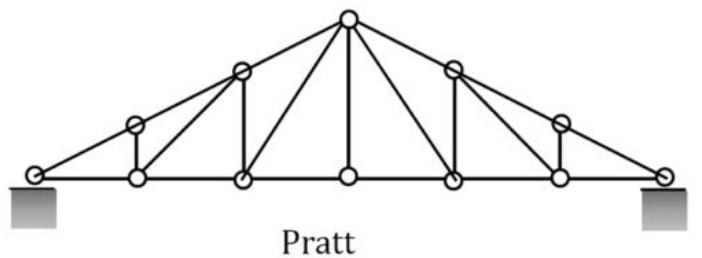
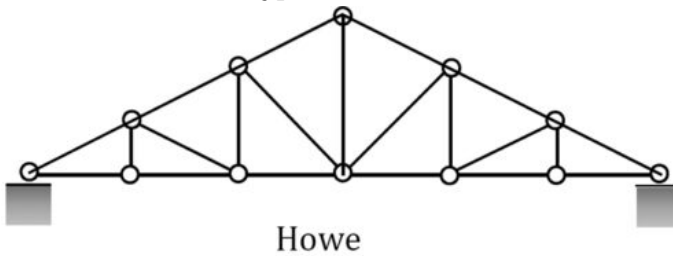
Here are common types of bridge trusses:



Source:

[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.05%3A\\_Internal\\_Forces\\_in\\_Plane\\_Trusses](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.05%3A_Internal_Forces_in_Plane_Trusses)

Here are common types of roof trusses:



Source:

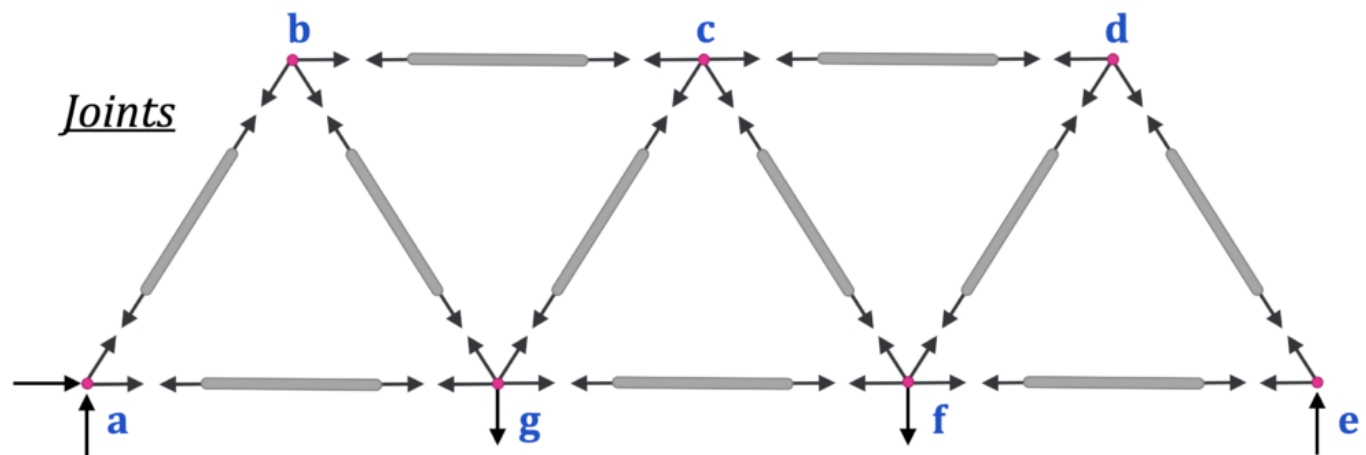
[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.05%3A\\_Internal\\_Forces\\_in\\_Plane\\_Trusses](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.05%3A_Internal_Forces_in_Plane_Trusses)

## 5.1.3 Parts of a Truss

A truss is composed of:

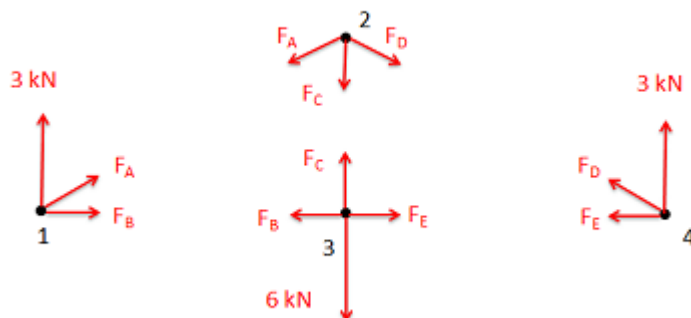
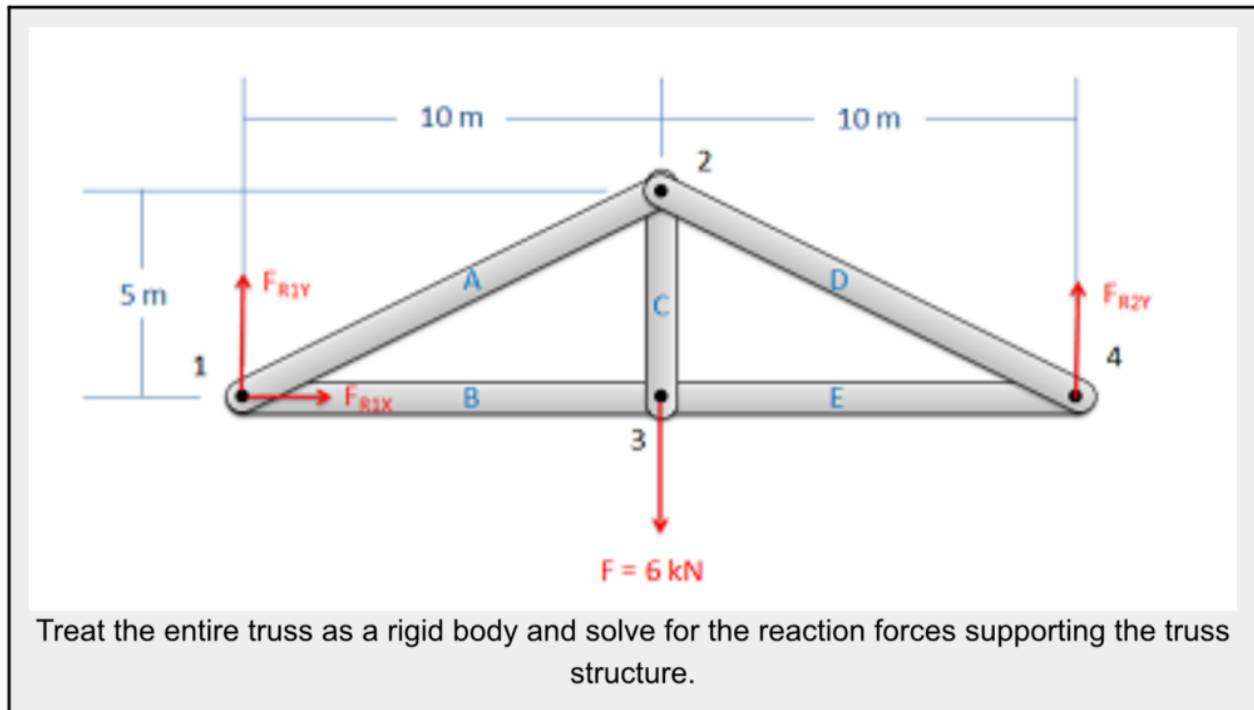
- joints
- members, and
- external forces (reaction forces and applied forces).

The ***joints*** are often labelled with a letter and are where the external forces and members connect.



Here is an example of just the joints without the members:

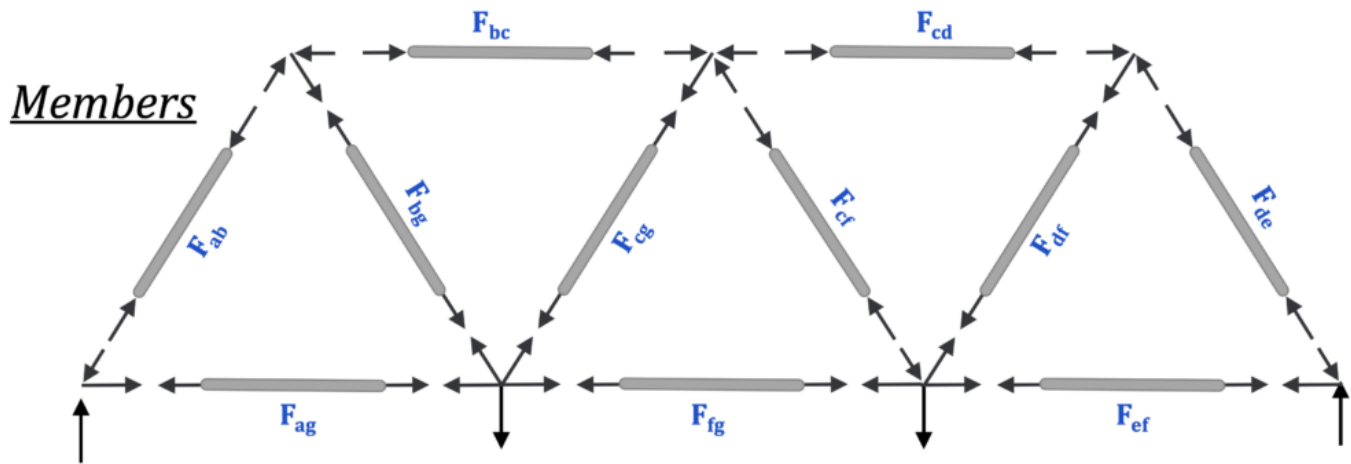




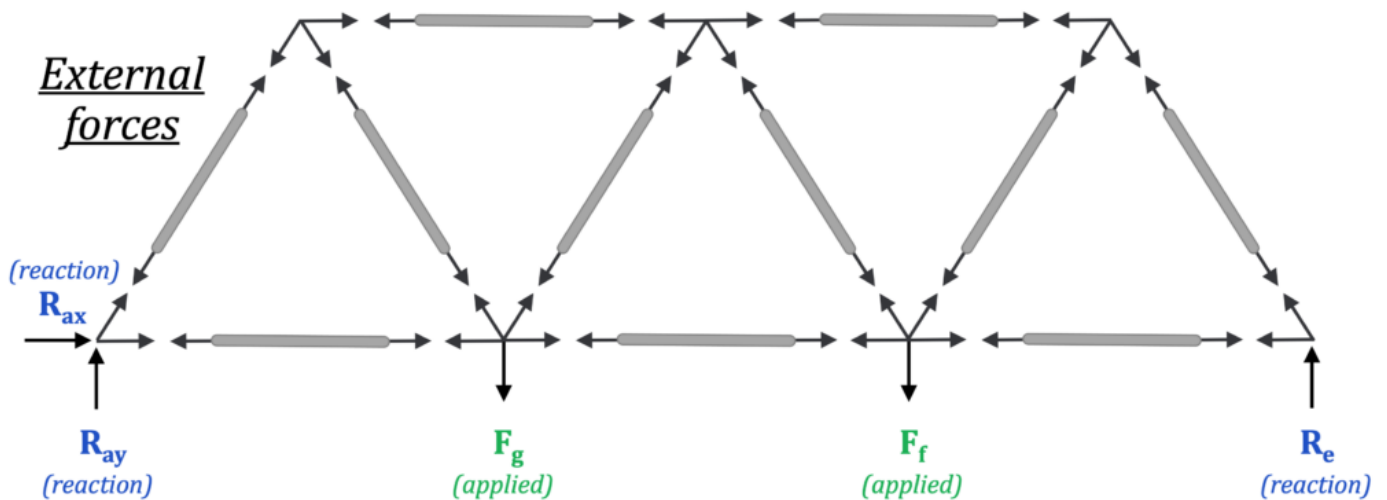
Source: Engineering Mechanics, Jacob

Moore, et al. [http://mechanicsmap.psu.edu/websites/5\\_structures/5-4\\_method\\_of\\_joints/methodofjoints.html](http://mechanicsmap.psu.edu/websites/5_structures/5-4_method_of_joints/methodofjoints.html)

The **members** are the metal or wooden beams that are labelled with the connection between joints. For example member AB connects joints A and B.



The **external forces** are the reaction forces and the applied forces. The applied forces come from the load distributed across the bridge or from the roof.



The applied force / load from trucks and cars goes from the deck, to the stringers, across the beams, to the joints of the truss where it is carried as applied (external) forces on the edges of the bridge.

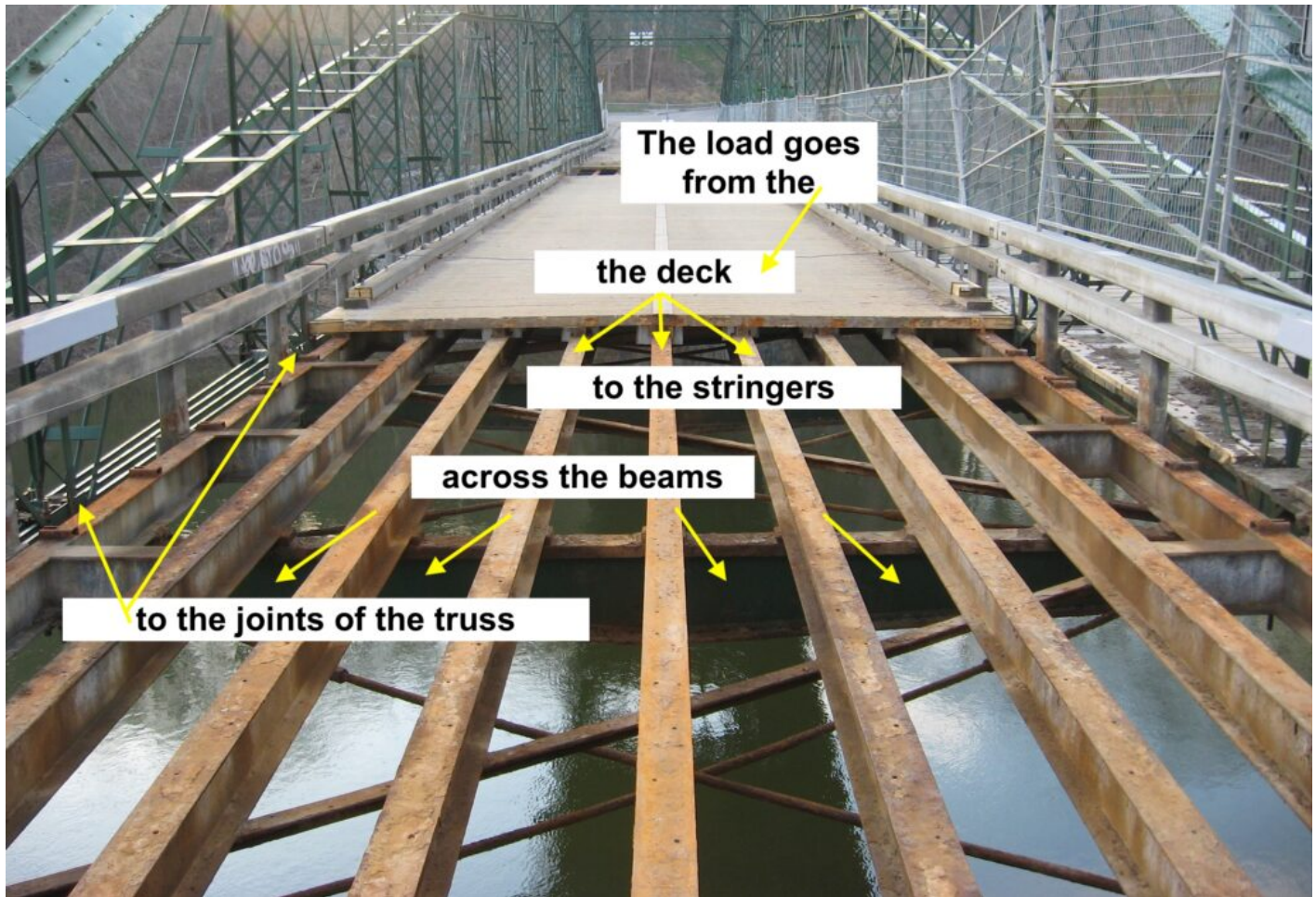


Image annotated from original source:

[https://upload.wikimedia.org/wikipedia/en/2/25/Nine\\_stringers%2C\\_2\\_floorbeams.jpg](https://upload.wikimedia.org/wikipedia/en/2/25/Nine_stringers%2C_2_floorbeams.jpg)

Here is a second type of structure. Which are the *stringers* and which are the *beams*?



Annotations added from original image source:

<https://www.maxpixel.net/static/photo/1x/Buildings-Leaves-Park-Autumn-Road-Fall-Structure-5623840.jpg>

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Any ideas?

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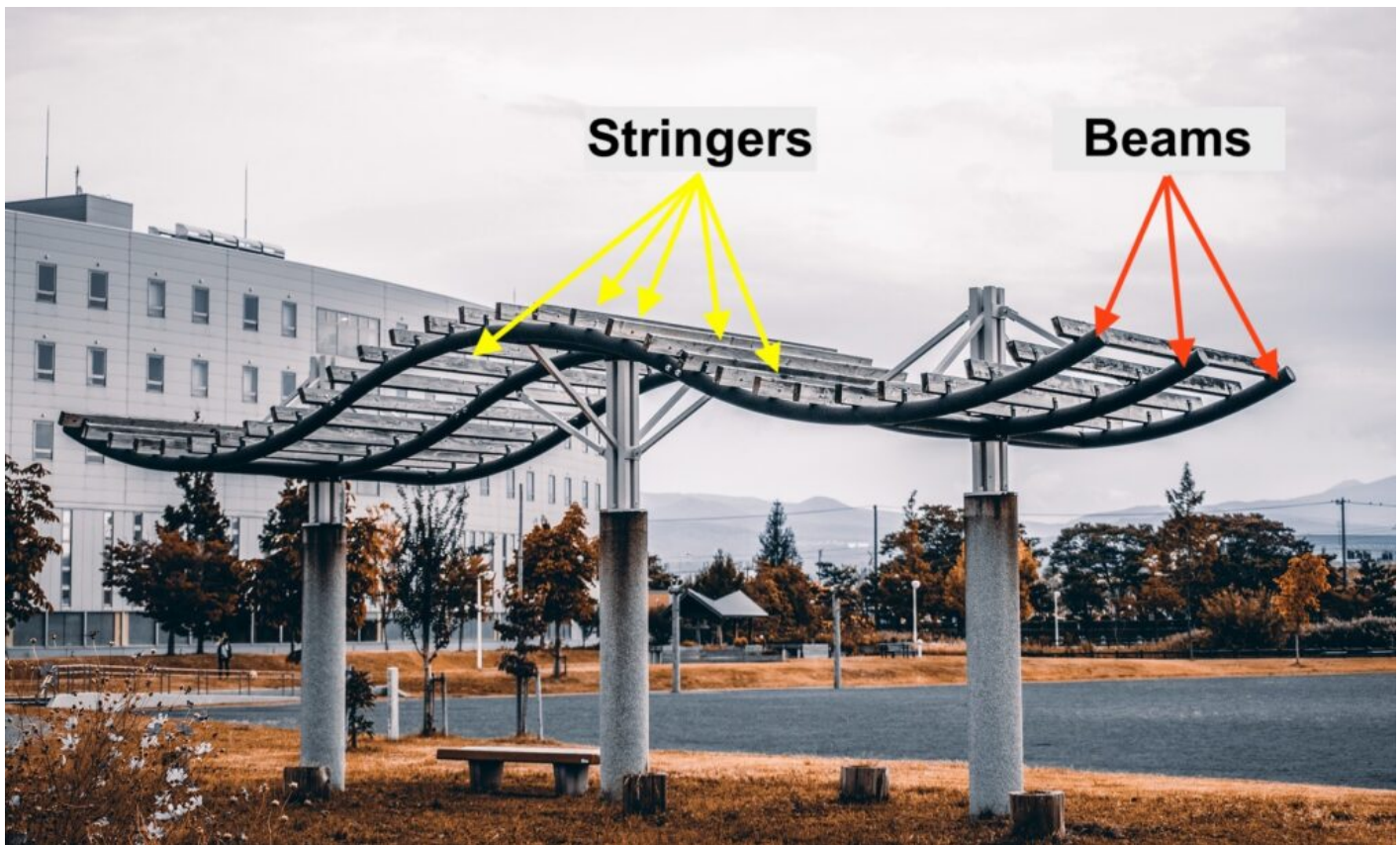
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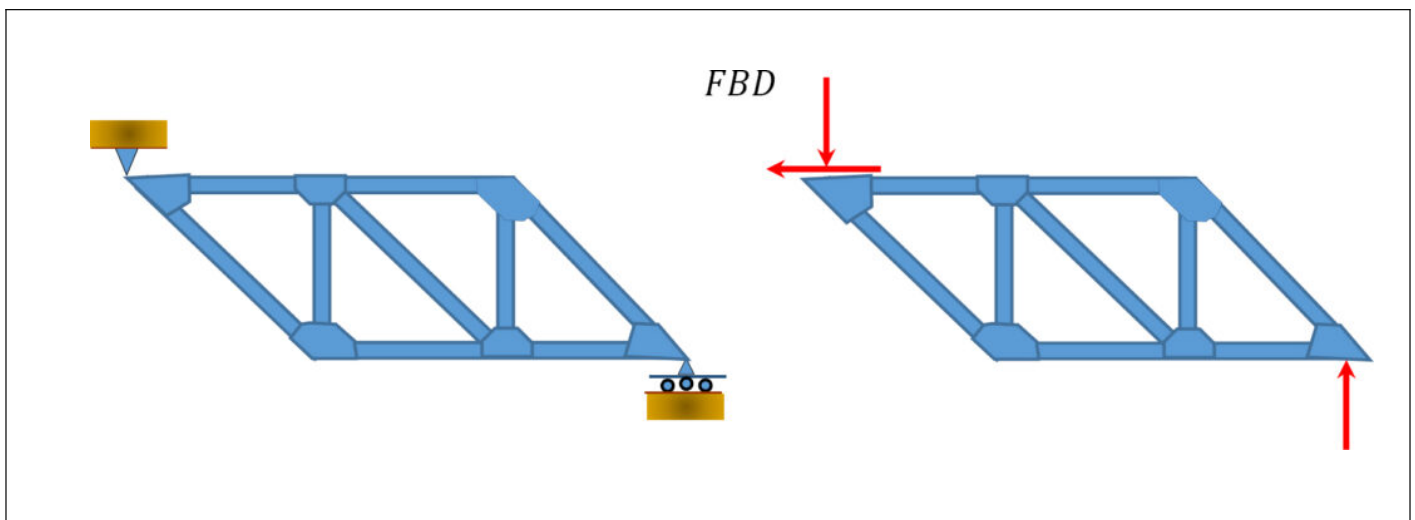
Here's the answer!

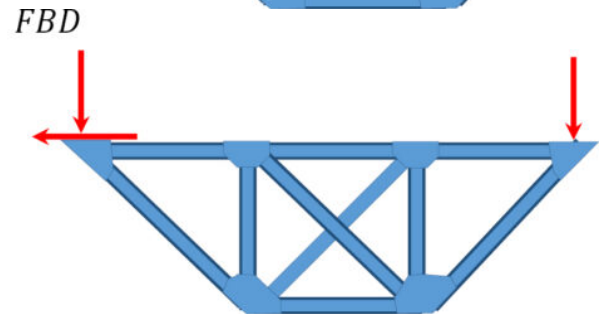
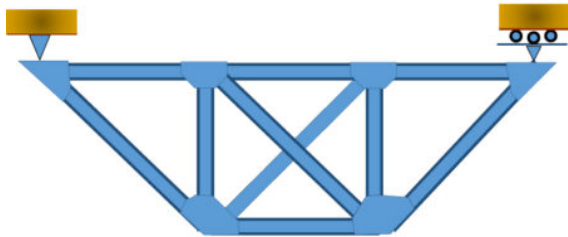
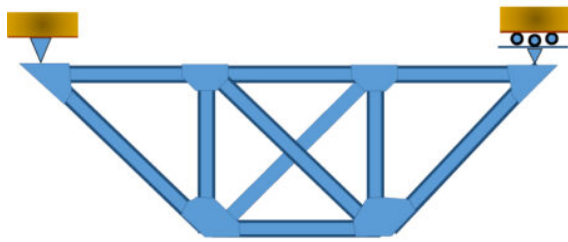
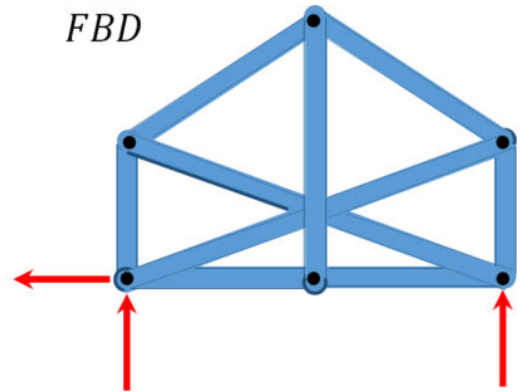
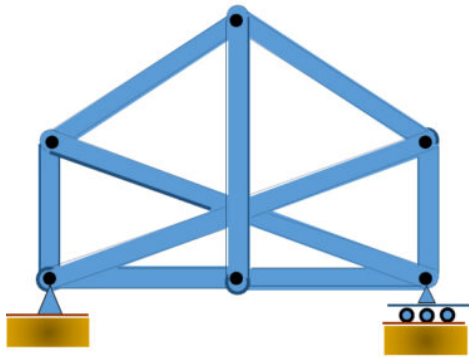


Annotations added from original source:

<https://www.maxpixel.net/static/photo/1x/Buildings-Leaves-Park-Autumn-Road-Fall-Structure-5623840.jpg>

Here are some examples on how to convert the reaction forces / moments for a truss. Note: these are the same as in section 3.4.



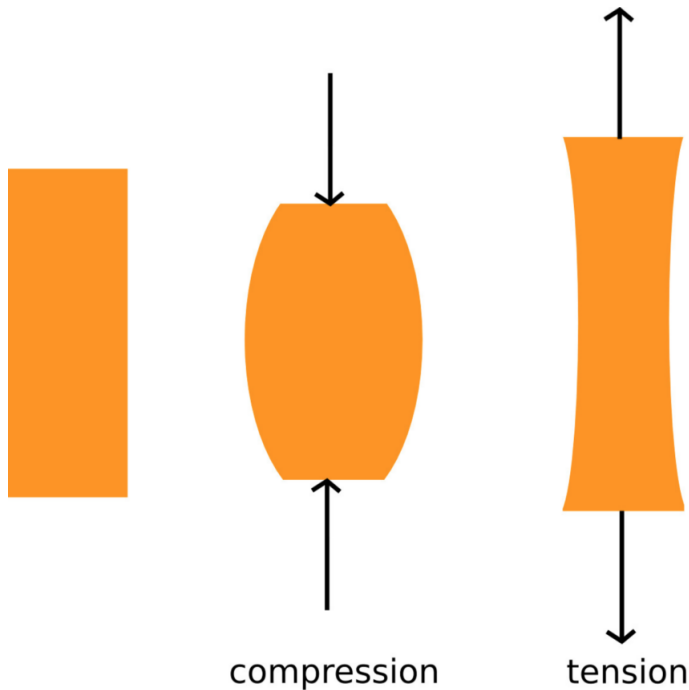


Source: Internal Forces in Beams and Frames, Libretexts.

[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.05%3A\\_Internal\\_Forces\\_in\\_Plane\\_Trusses](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.05%3A_Internal_Forces_in_Plane_Trusses)

## 5.1.4 Tension & Compression

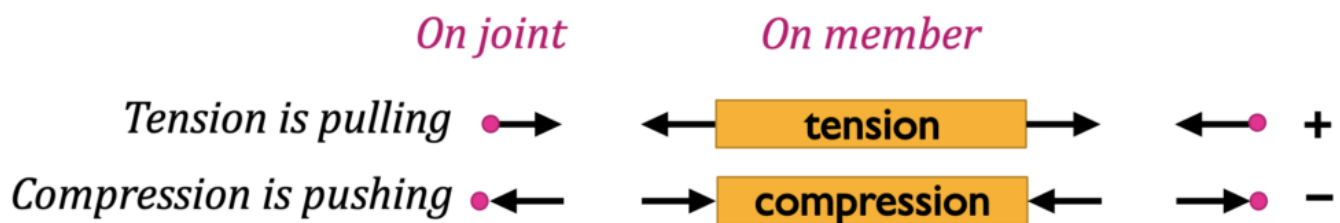
The two-force members carry internal forces in either tension or compression between the joints. One standard sign convention is to assume all members are in tension, labelled as positive (+), then any negative number (-) means the member is in compression.



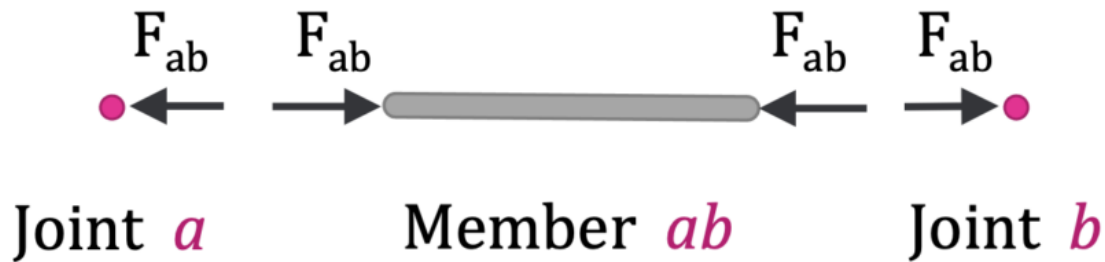
Source:

<https://www.flickr.com/photos/121935927@N06/13580545445>

Following Newton's 3rd law regarding equal and opposite reactions, when there is tension in a member, there is also tension in a joint. Pulling on the member (tension) in turn pulls on the joint. Similarly, pushing on a member (compression) pushes on the joint as well.

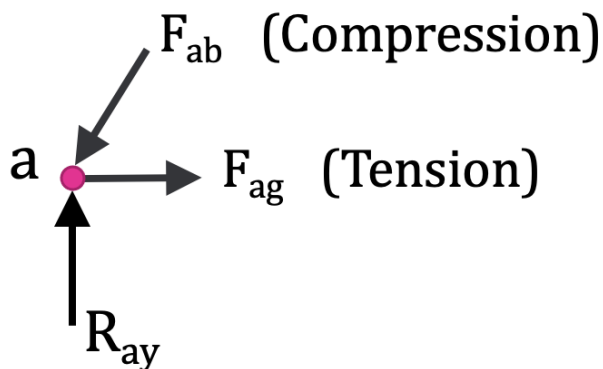


Similarly, the force from member AB ( $F_{ab}$ ) is distributed from joint a through member ab to joint b. Shown here in compression,  $F_{ab}$  is negative. The magnitude of  $F_{ab}$  on joint a is the same as the magnitude on joint b, even though they are pointing two different directions (hence equal and opposite). Member bc will have a different magnitude.



When you look at each joint, compression (-) appears to be pushing on the joint while tension (+) is pulling on it with the force named for the member (  $F_{ab}$  ).

*Joint a:*



In the next section, we will discuss each of these methods in greater detail and how to solve problems using them.

#### Key Takeaways

*Basically:* A truss is a rigid structure composed of two force members (where forces are applied at only two locations) that connect at joints and have external forces applied. The internal forces of the truss put members in compression (-) or tension (+).

*Application:* The frame of a roof is often composed of a wooden truss, and trusses are commonly found in wooden and metal bridges.

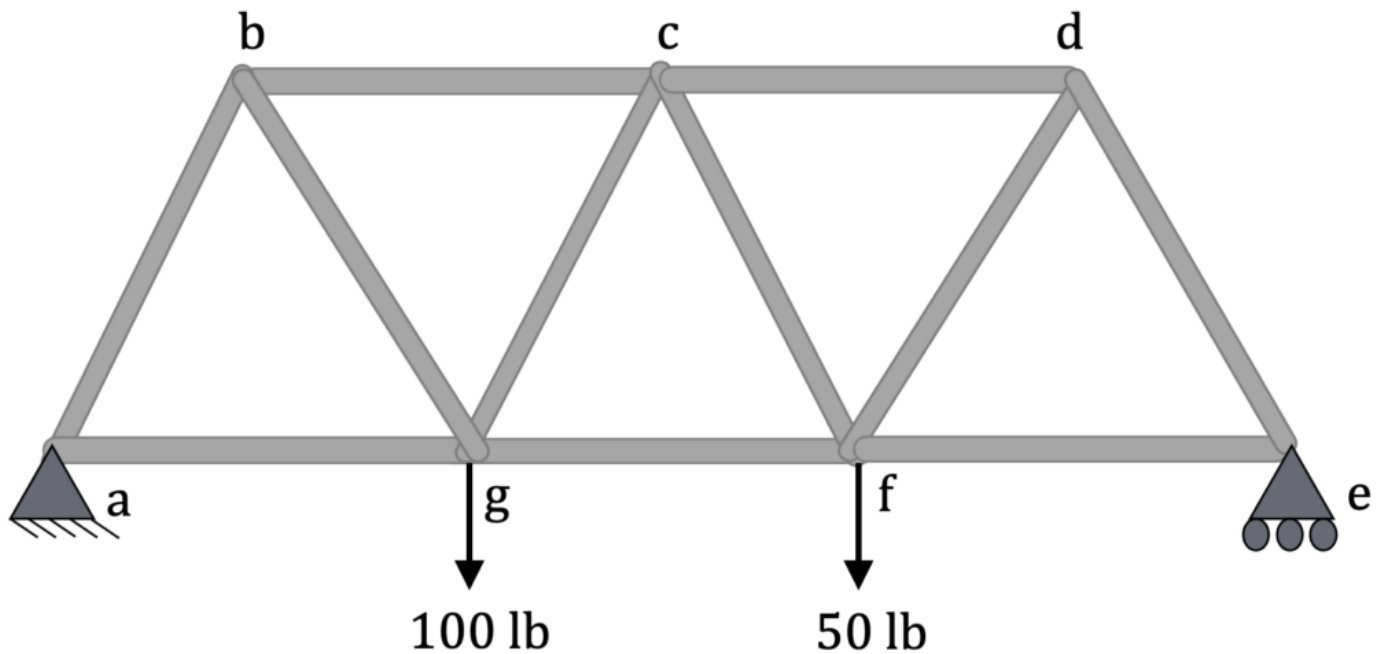
*Looking Ahead:* The next two sections discuss the method for calculating the force in the members & you'll talk about trusses more in your Structures course.



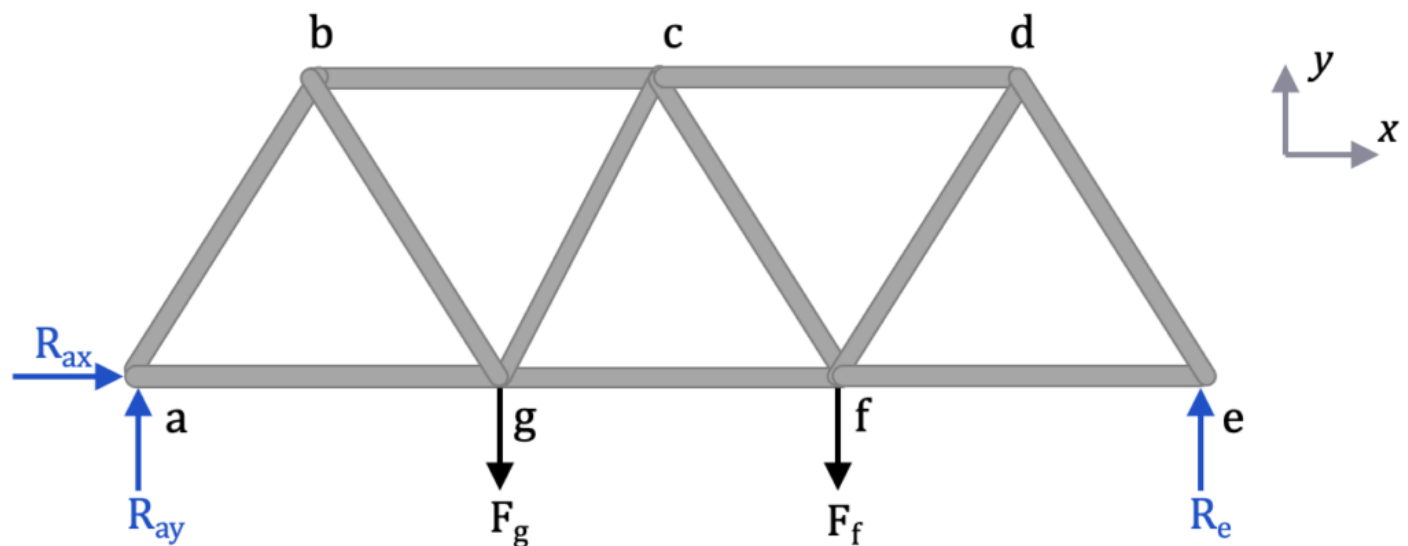


## 5.2 Method of Joints

The **method of joints** is a form of particle analysis. After solving for the reaction forces, you solve for the unknown forces at each joint until you have found the value of each member. You start with your model:



Convert the constraints into reaction forces with the appropriate labels:



Now solve for the reaction forces ( $R_{ax}$   $R_{ay}$   $R_e$ ) looking only at the external forces using the equilibrium equations for a rigid body:

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M = 0$$

Assuming the length of each member is  $L$ :

$$\sum F_x = R_{ax} = 0, \quad \underline{R_{ax} = 0}$$

$$\sum F_y = R_{ay} + R_e - F_g - F_f = 0, \quad R_{ay} + R_e = 150 \text{ lb}$$

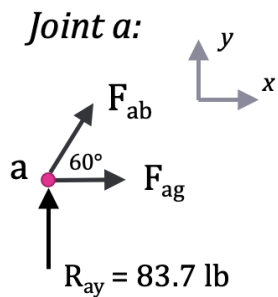
$$\sum M_a = -L \cdot F_g - 2L \cdot F_f + 3L \cdot R_e = 0 \quad R_e = \frac{100L + 100L}{3L} \quad \underline{R_e = 66.7 \text{ lb}}$$

$$R_{ay} = 150 \text{ lb} - 66.7 \text{ lb} \quad \underline{R_{ay} = 83.3 \text{ lb}}$$

Next, pick a joint where there are 2 or fewer unknown values such as  $a$  or  $e$ . This is because you only have 2 equations available to find the unknowns:  $\sum F_x = 0$  and  $\sum F_y = 0$ . The following table shows the number of known and unknown forces at each joint.

Joint:  
Known forces:  
Unknown forces:

Choosing joint a (or e), do a particle analysis, assuming all of the members are in tension. That way, if the force is negative, that means it is in compression. Notice  $R_{ax}$  has been excluded because it is equal to zero.

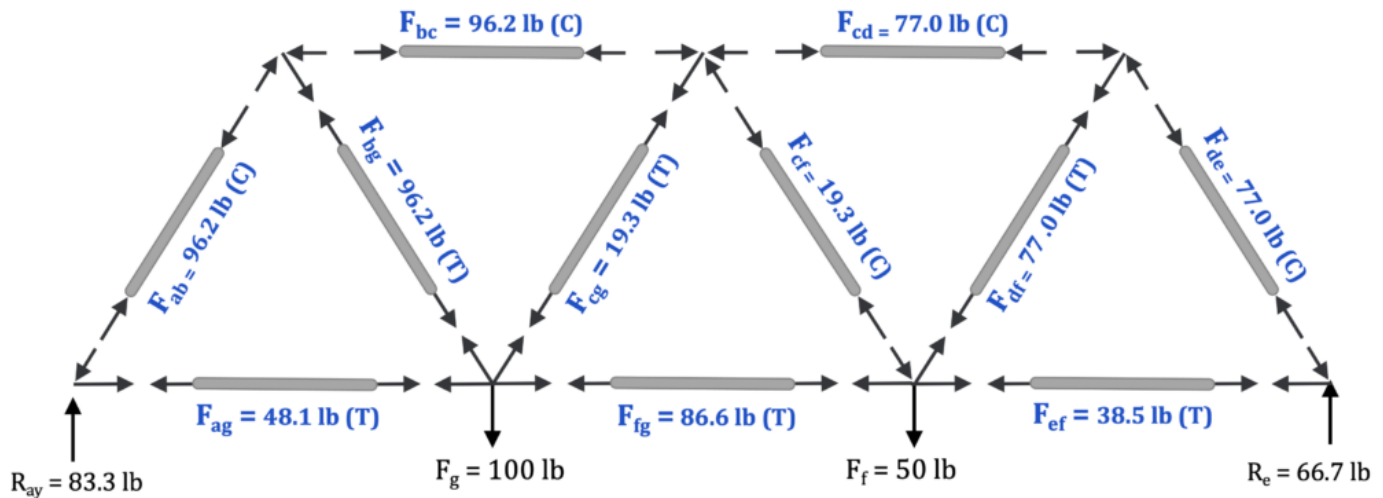


$$\sum F_y = 0 \Rightarrow R_{ay} + F_{ab} \sin(60^\circ) = 0 \Rightarrow F_{ab} = -\frac{R_{ay}}{\sin(60^\circ)} = -\frac{83.7}{0.866} \Rightarrow \underline{F_{ab} = -96.2 \text{ lb}} \Rightarrow \text{(compression)}$$

$$\sum F_x = 0 \Rightarrow F_{ag} + F_{ab} \cos(60^\circ) = 0 \Rightarrow F_{ag} = -F_{ab} \cos(60^\circ) = -(-96.2 \text{ lb}) * (0.5) \Rightarrow \underline{F_{ag} = +48.1 \text{ lb}} \text{ (tension)}$$

Next move to joint b because you now only have 2 unknowns now at joint b ( $F_{bc}$  and  $F_{bg}$ ).

Keep analyzing joints until you've calculated the load in all members:



Member	ab	bc	cd	de	ef	fg	ag	bg	cg	cf	df
Force (lb)	96.2	96.2	77.0	77.0	38.5	86.6	48.1	96.2	19.3	19.3	77.0
Tension or Compression	C	C	C	C	T	T	T	T	T	C	T

And that's it! If you don't specify compression or tension, you should use positive and negative to denote tension and compression, respectively.

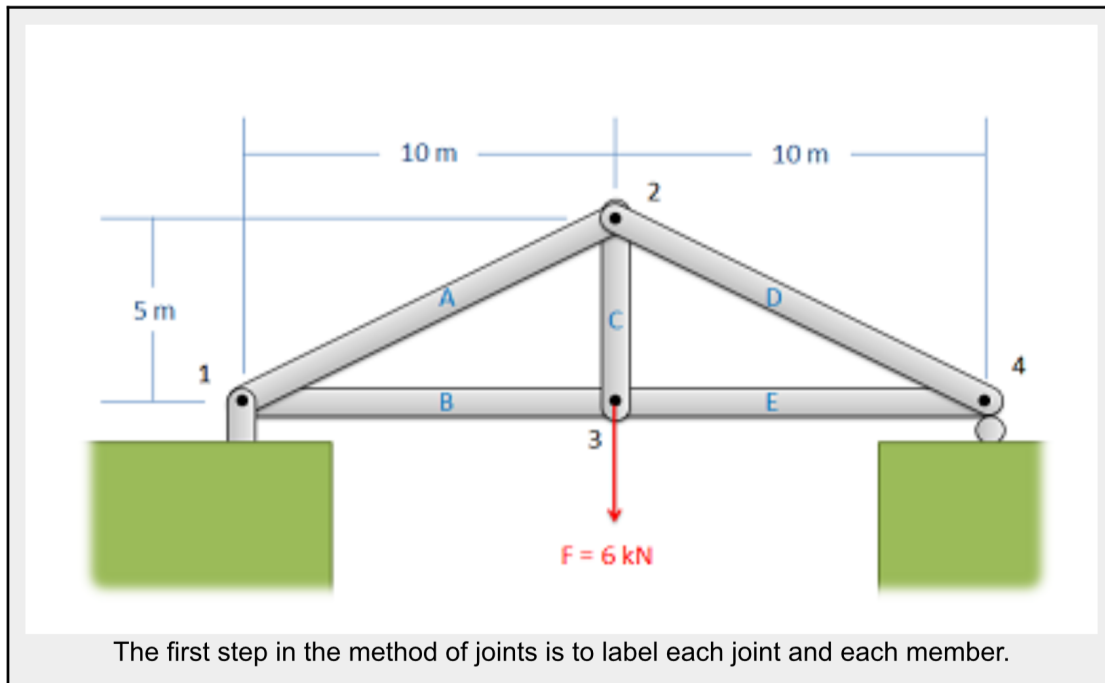
Here is a second explanation on how to solve using method of joints:

The **method of joints** is a process used to solve for the unknown forces acting on members of a **truss**. The method centers on the joints or connection points between the members, and it is usually the fastest and easiest way to solve for all the unknown forces in a truss structure.

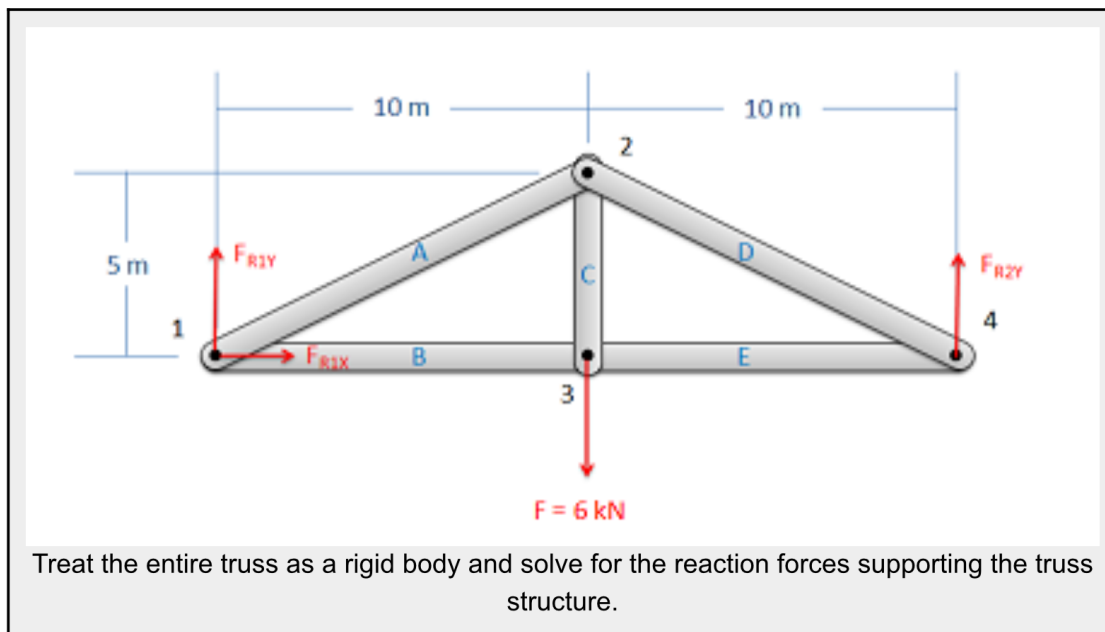
## Using This Method:

The process used in the method of joints is outlined below:

1. In the beginning it is usually useful to label the members and the joints in your truss. This will help you keep everything organized and consistent in later analysis. In this book, the members will be labeled with letters and the joints will be labeled with numbers.



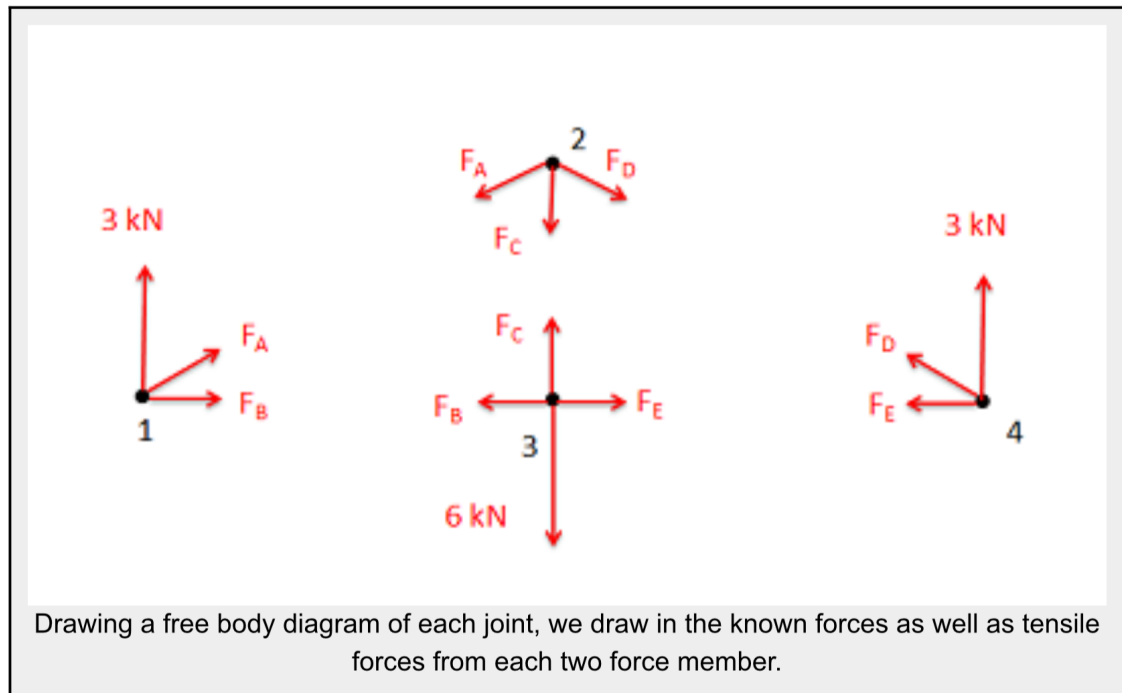
2. Treating the entire truss structure as a rigid body, draw a free body diagram, write out the equilibrium equations, and solve for the external reacting forces acting on the truss structure. This analysis should not differ from the analysis of a single rigid body.



3. Assume there is a pin or some other small amount of material at each of the connection points between the members. Next you will draw a free body diagram for each connection point. Remember to include:
  - Any external reaction or load forces that may be acting at that joint.
  - A normal force for each two force member connected to that joint. Remember that for a two force member, the force will be acting along the line between the two connection points on the member. We will also need to guess if it will be a tensile or a compressive force. An incorrect guess now though will

simply lead to a negative solution later on. A common strategy then is to assume all forces are tensile, then later in the solution any positive forces will be tensile forces and any negative forces will be compressive forces.

- Label each force in the diagram. Include any known magnitudes and directions and provide variable names for each unknown.



4. Write out the equilibrium equations for **each of the joints**. You should treat the joints as particles, so there will be force equations but no moment equations. This should give you a large number of equations.
  - The sum of the forces in the x direction will be zero and the sum of the forces in the y direction will be zero for each of the joints.  $\sum \vec{F} = 0 \implies \sum F_x = 0; \sum F_y = 0$
5. Finally, solve the equilibrium equations for the unknowns. You can do this algebraically, solving for one variable at a time, or you can use matrix equations to solve for everything at once. If you assumed that all forces were tensile earlier, remember that negative answers indicate compressive forces in the members.

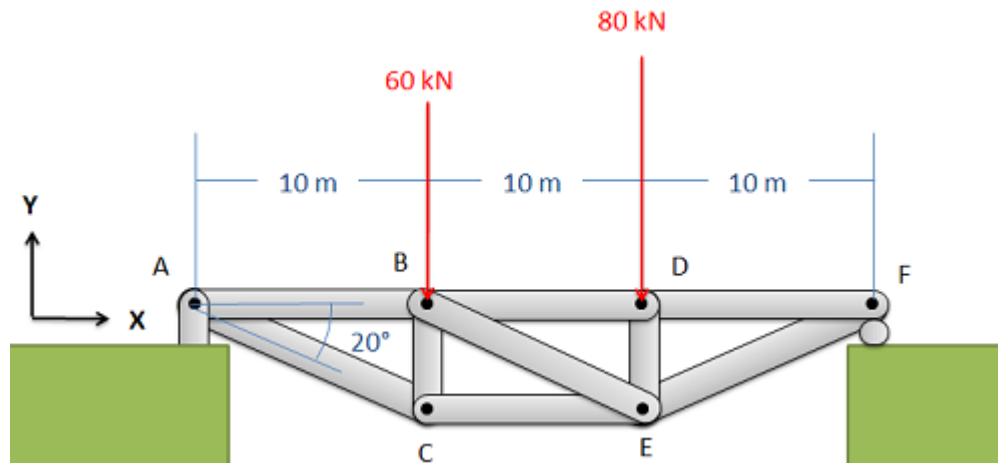
Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/5\\_structures/5-4\\_method\\_of\\_joints/methodofjoints.html](http://mechanicsmap.psu.edu/websites/5_structures/5-4_method_of_joints/methodofjoints.html)

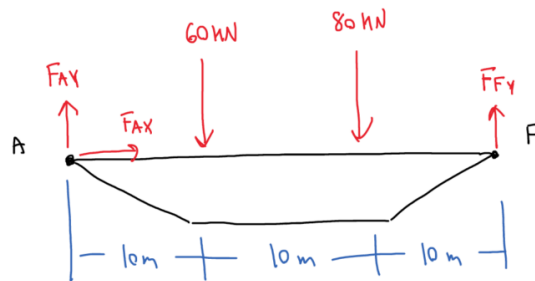
Additional examples from the [Engineering Mechanics webpage](#):

### Example 1:

Find the force acting in each of the members in the truss bridge shown below. Remember to specify if each member is in tension or compression.



Solution:



$$\sum F_x = F_{AX} = 0$$

$$\sum F_y = F_{AY} + F_{FY} - 60 - 80 = 0$$

$$\sum M_A = (F_{FY})(30) - (60)(10) - (80)(20) = 0$$

$$F_{FY} = \frac{(60)(10) + (80)(20)}{30} = 73.33 \text{ kN}$$

$$F_{AY} = 60 + 80 - 73.33 = 66.67 \text{ kN}$$

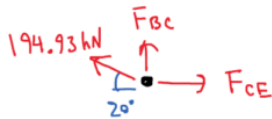


Point A



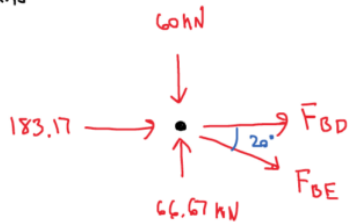
$$\begin{aligned}\sum F_x &= F_{AB} + F_{AC} \cos(20) = 0 \\ \sum F_y &= 66.67 - F_{AC} \sin(20) = 0 \\ F_{AC} &= \frac{66.67}{\sin(20)} = \underline{194.93 \text{ kN}} \\ F_{AB} &= -\cos(20)(194.93) = \underline{-183.17 \text{ kN}}\end{aligned}$$

Point C



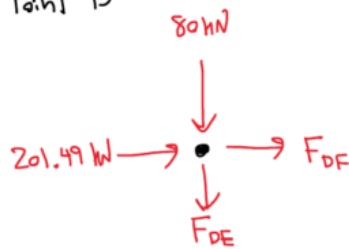
$$\begin{aligned}\sum F_x &= -194.93 \cos(20) + F_{CE} = 0 \\ \sum F_y &= 194.93 \sin(20) + F_{BC} = 0 \\ F_{CE} &= 194.93 \cos(20) = \underline{183.17 \text{ kN}} \\ F_{BC} &= -194.93 \sin(20) = \underline{-66.67 \text{ kN}}\end{aligned}$$

Point B



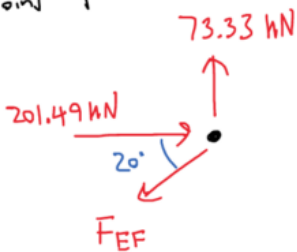
$$\begin{aligned}\sum F_x &= 183.17 + F_{BD} + F_{BE} \cos(20) = 0 \\ \sum F_y &= 66.67 - 60 - F_{BE} \sin(20) = 0 \\ F_{BE} &= \frac{6.67}{\sin(20)} = \underline{19.50 \text{ kN}} \\ F_{BD} &= -183.17 - (19.50) \cos(20) = \underline{-201.49 \text{ kN}}\end{aligned}$$

Point D



$$\begin{aligned}\sum F_x &= 201.49 + F_{DF} = 0 \\ \sum F_y &= -80 - F_{DE} = 0 \\ F_{DF} &= -201.49 \text{ kN} \\ F_{DE} &= -80 \text{ kN}\end{aligned}$$

Point F



$$\begin{aligned}\sum F_y &= 73.33 - F_{EF} \sin(20^\circ) = 0 \\ F_{EF} &= \frac{73.33}{\sin(20^\circ)} = 214.40 \text{ kN}\end{aligned}$$

Solution:

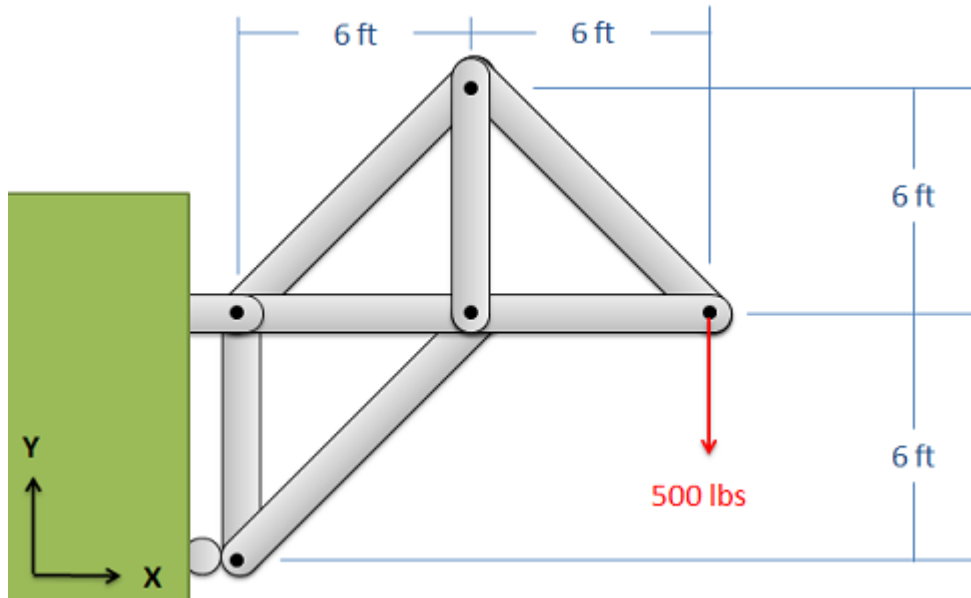
$$\begin{aligned}F_{AB} &= 183.17 \text{ kN C} \\ F_{AC} &= 194.93 \text{ kN T} \\ F_{BC} &= 66.67 \text{ kN C} \\ F_{BD} &= 201.49 \text{ kN C} \\ F_{BE} &= 19.50 \text{ kN T} \\ F_{CE} &= 183.17 \text{ kN T} \\ F_{DE} &= 80 \text{ kN C} \\ F_{DF} &= 201.49 \text{ kN C} \\ F_{EF} &= 214.40 \text{ kN T}\end{aligned}$$

Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/5\\_structures/5-4\\_method\\_of\\_joints/pdf/MethodOfJoints\\_WorkedExample1.pdf](http://mechanicsmap.psu.edu/websites/5_structures/5-4_method_of_joints/pdf/MethodOfJoints_WorkedExample1.pdf)

**Example 2:**

Find the force acting in each of the members of the truss shown below. Remember to specify if each member is in tension or compression.



Solution here.

In summary:

## Method of Joints

### 1) Solve for reactions using system Eq Eq's

$$\sum \underline{F_x} = 0 \quad \sum \underline{F_y} = 0, \quad \sum M = 0$$

- **Draw system FBD**
- **Sum forces in X, Y**
- **Sum moments about point with most unknown reaction forces**



### 2) Solve for each joint summing forces in only x and y (no moment)

- **Draw a FBD at each joint**
- **Sum forces in X and Y at each joint**

$$\sum \underline{F_x} = 0 \quad \sum \underline{F_y} = 0$$

- **Pick a joint where you have at least 1 known and no more than 2 unknown forces**
- **If you assume all forces are in tension, then if it comes out negative, it's in compression**



#### Key Takeaways

*Basically:* Method of joints is an analysis technique to find the forces in the members of a truss. It looks at each joint individually using the particle equilibrium equations.

*Application:* To calculate the loads on bridges and roofs, especially if you need to know all of the values of the members.

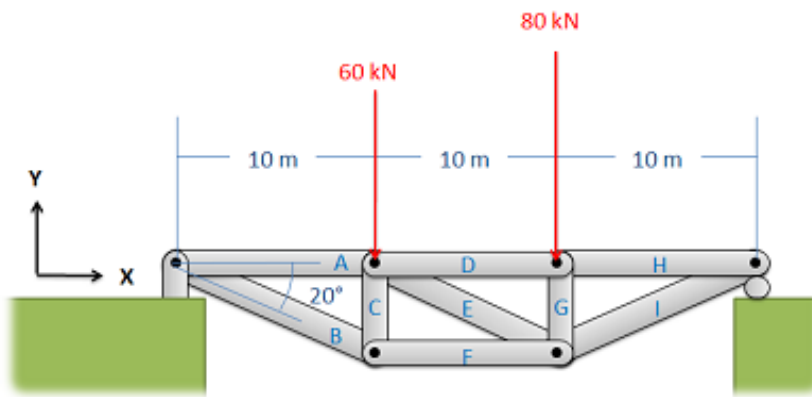
*Looking Ahead:* The next section explores a method to solve one or two members of a truss (instead of finding all of them).



## 5.3 Method of Sections

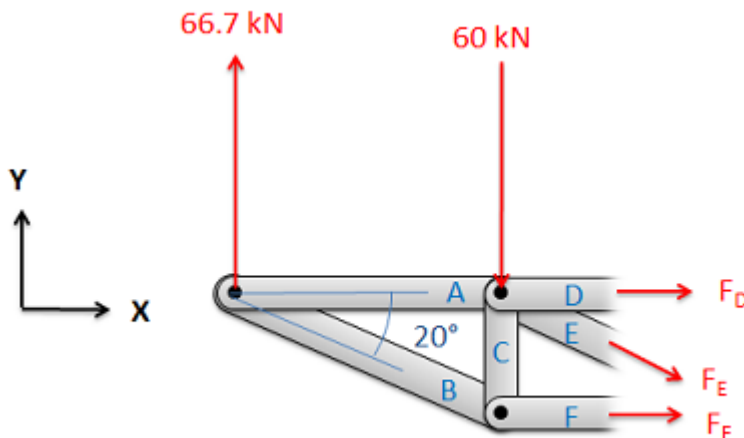
The **method of sections** uses rigid body analysis to solve for a specific member or two. Instead of looking at each joint, you make a cut through the truss, turning the members along that line into internal forces (assume in tension). Then you solve the rigid body using the equilibrium equations for a rigid body:  $\sum F_x = 0$ ;  $\sum F_y = 0$ ;  $\sum M_z = 0$

The truss:



Source: Engineering Mechanics, Jacob Moore, et al.  
[http://mechanicsmap.psu.edu/websites/5\\_structures/5-5\\_method\\_of\\_sections/methodofsections.html](http://mechanicsmap.psu.edu/websites/5_structures/5-5_method_of_sections/methodofsections.html)

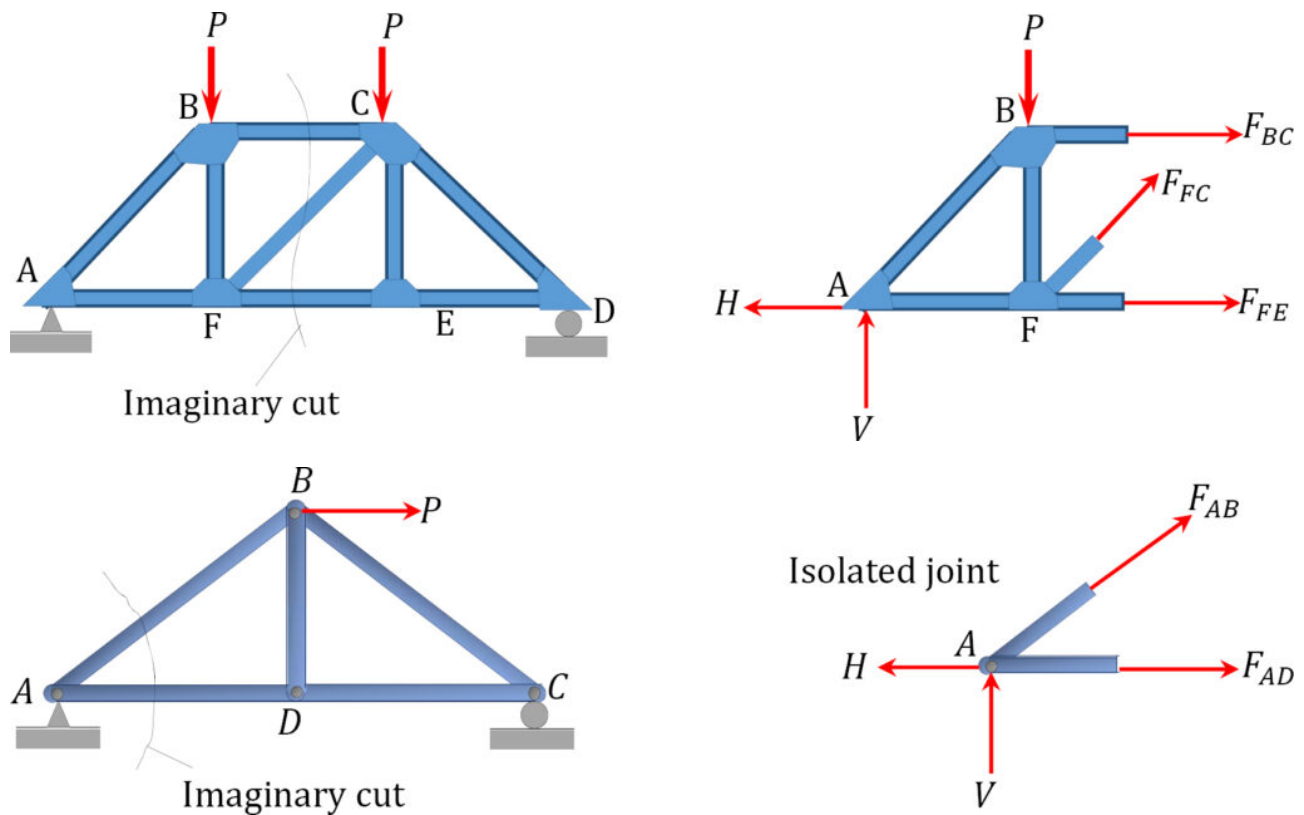
is split into two to solve for  $F_E$ .

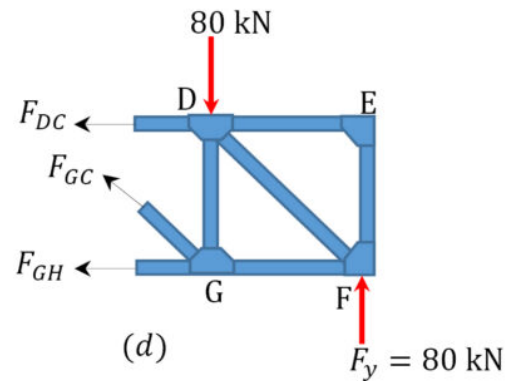
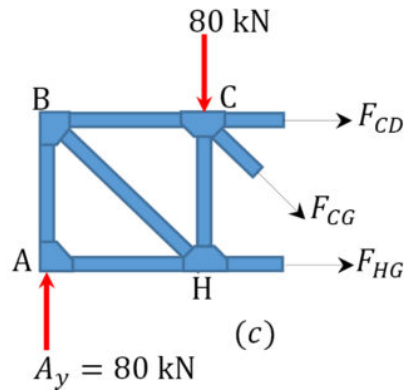
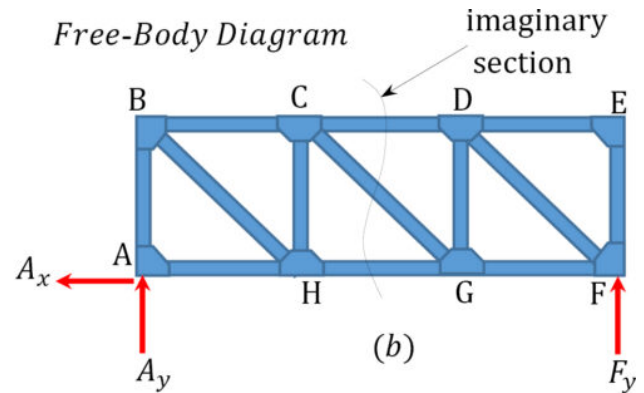
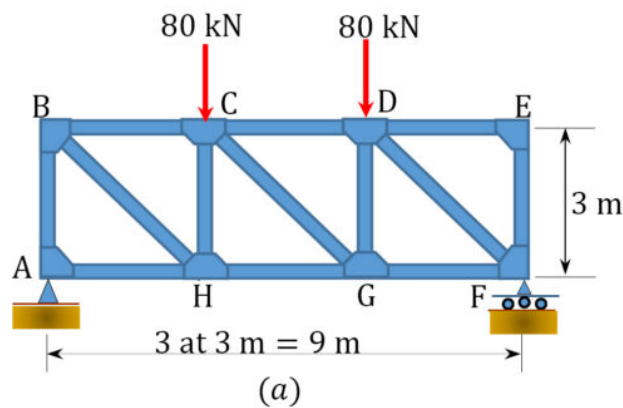


Source: Engineering Mechanics, Jacob Moore, et al.  
[http://mechanicsmap.psu.edu/websites/5\\_structures/5-5\\_method\\_of\\_sections/methodofsections.html](http://mechanicsmap.psu.edu/websites/5_structures/5-5_method_of_sections/methodofsections.html)

For this example, you could choose the right half or left half. For some problems, being strategic is necessary otherwise you need to make multiple cuts. In this problem you had to solve for the reaction forces first, but that isn't always the case as you can sometimes just make the cut (see example 2 below).

Here are more examples of how to make a cut and showing the naming convention:





Source: Internal Forces in Beams and Frames, Libretexts.

[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.05%3A\\_Internal\\_Forces\\_in\\_Plane\\_Trusses](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.05%3A_Internal_Forces_in_Plane_Trusses)

Here is a detailed explanation:

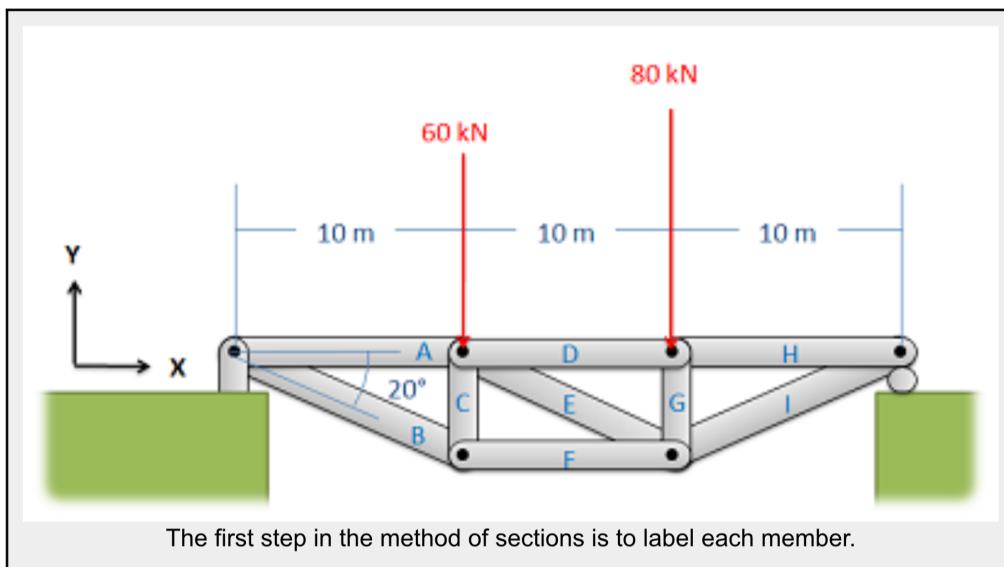
The **method of sections** is a process used to solve for the unknown forces acting on members of a **truss**. The method involves breaking the truss down into individual sections and analyzing each section as a separate rigid body. The method of sections is usually the fastest and easiest way to determine the unknown forces acting in a specific member of the truss.

## Using This Method:

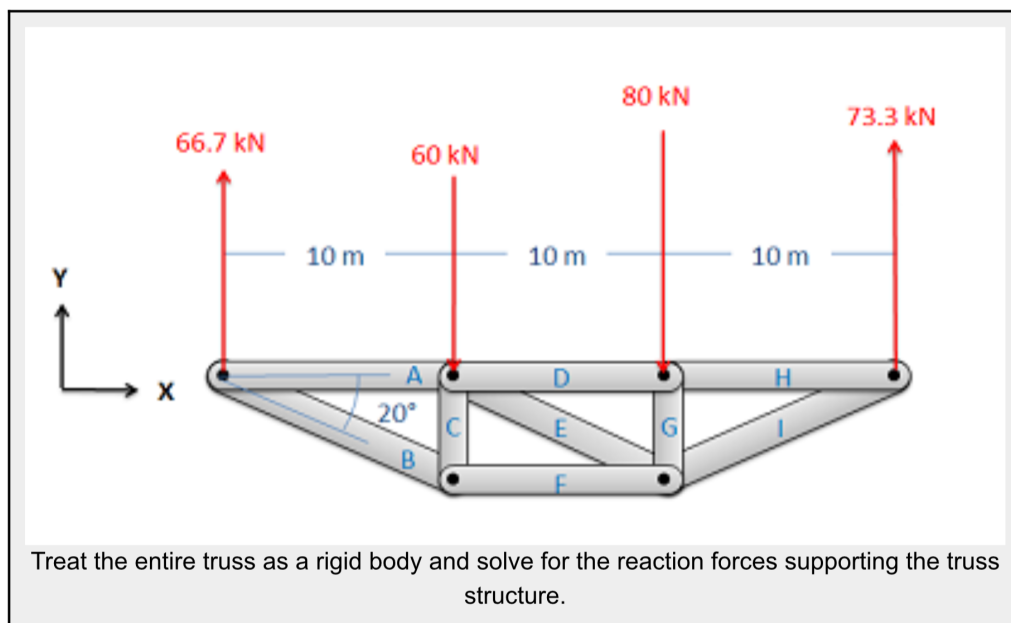
The process used in the method of sections is outlined below:

1. In the beginning it is usually useful to label the members in your truss. This will help you keep everything organized and consistent in later analysis. In this book, the members will be labeled with letters.

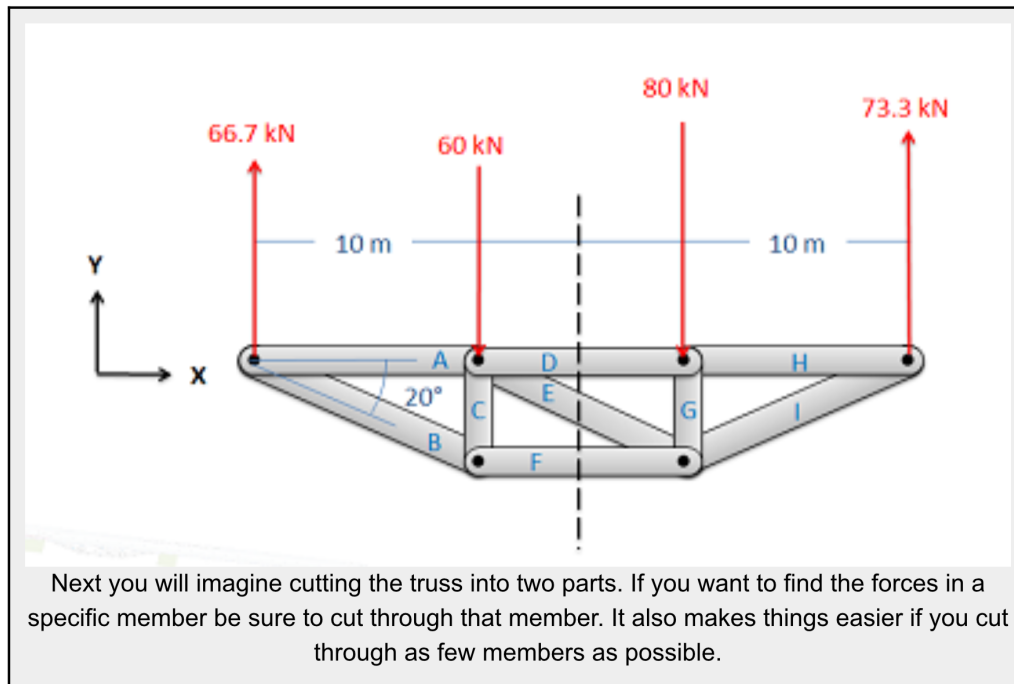




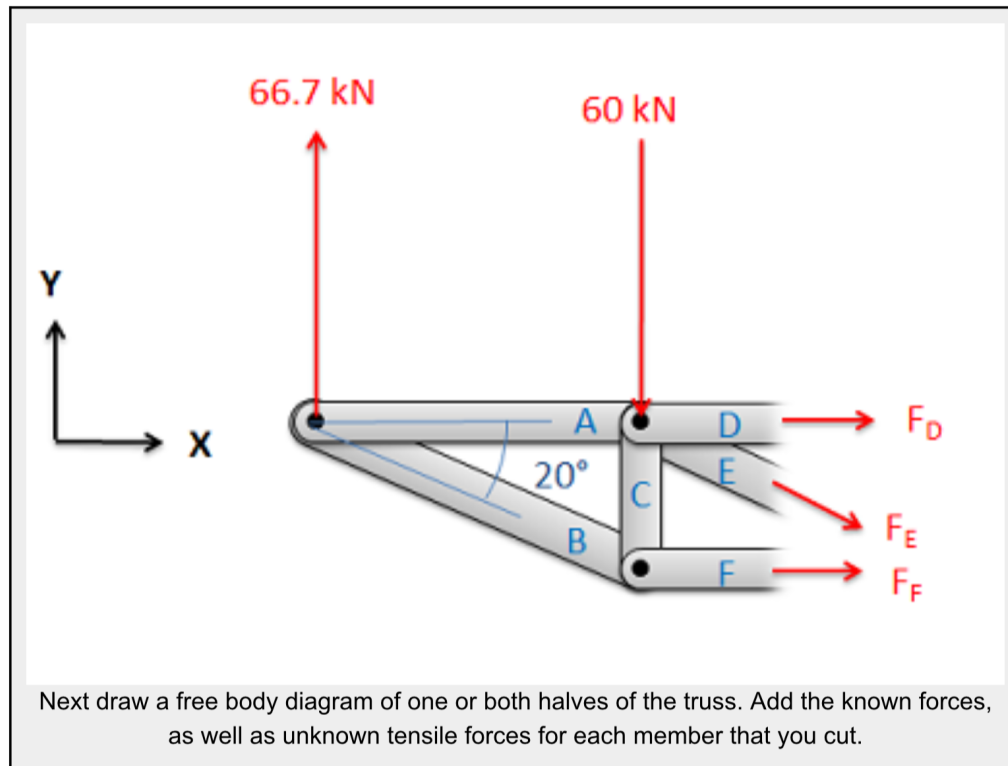
2. Treating the entire truss structure as a rigid body, draw a free body diagram, write out the equilibrium equations, and solve for the external reacting forces acting on the truss structure. This analysis should not differ from the analysis of a single rigid body.



3. Next you will imagine cutting your truss into two separate sections. The cut should travel through the member that you are trying to solve for the forces in, and should cut through as few members as possible (The cut does not need to be a straight line).



4. Next you will draw a free body diagram for either one, or both sections that you created. Be sure to include all the forces acting on each section.
  - Any external reaction or load forces that may be acting at the section.
  - An internal force in each member that was cut when splitting the truss into sections. Remember that for a two force member, the force will be acting along the line between the two connection points on the member. We will also need to guess if it will be a tensile or a compressive force. An incorrect guess now though will simply lead to a negative solution later on. A common strategy then is to assume all forces are tensile, then later in the solution any positive forces will be tensile forces and any negative forces will be compressive forces.
  - Label each force in the diagram. Include any known magnitudes and directions and provide variable names for each unknown.



5. Write out the equilibrium equations for each section you drew a free body diagram of. These will be extended bodies, so you will need to write out the force and the moment equations.
  - You will have three possible equations for each section, two force equations and one moment equation.  $\sum \vec{F} = 0$ ;  $\sum \vec{M} = 0$ ;  $\sum F_x = 0$ ;  $\sum F_y = 0$ ;  $\sum M_z = 0$
6. Finally, solve the equilibrium equations for the unknowns. You can do this algebraically, solving for one variable at a time, or you can use matrix equations to solve for everything at once. If you assumed that all forces were tensile earlier, remember that negative answers indicate compressive forces in the members.

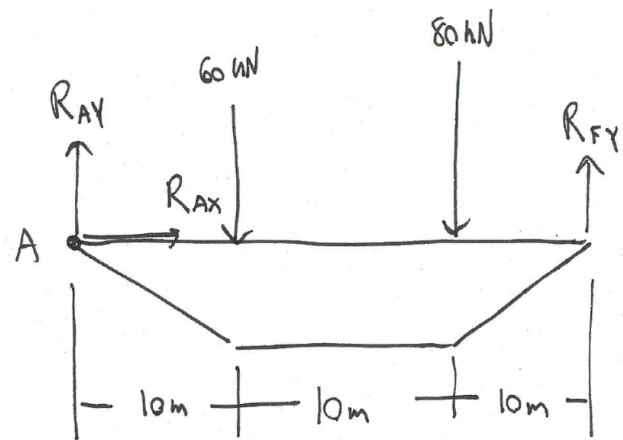
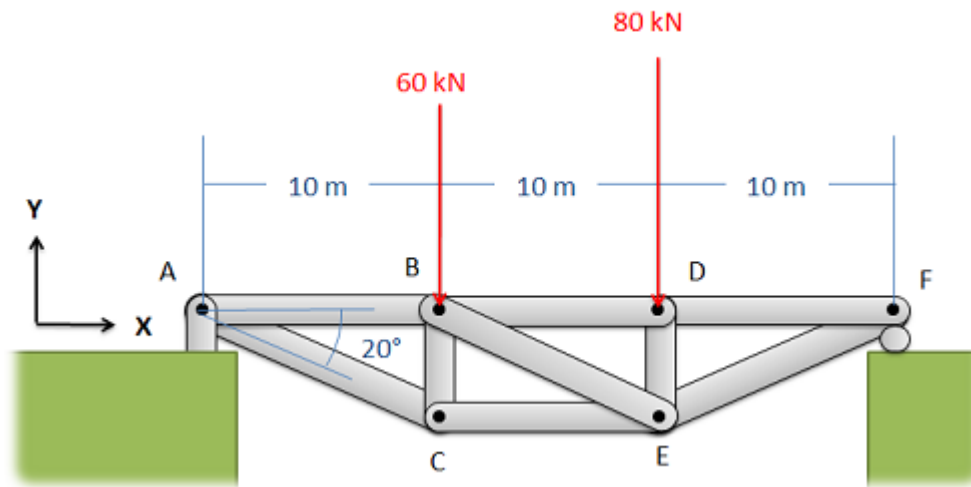
Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/5\\_structures/5-5\\_method\\_of\\_sections/methodofsections.html](http://mechanicsmap.psu.edu/websites/5_structures/5-5_method_of_sections/methodofsections.html)

Additional examples from the [Engineering Mechanics](#) webpage:

### Example 1:

Find the forces acting on members BD and CE. Be sure to indicate if the forces are tensile or compressive.



$$\sum F_x = R_{AX} = 0$$

$$\sum F_y = R_{AY} + R_{FY} - 60 - 80 = 0$$

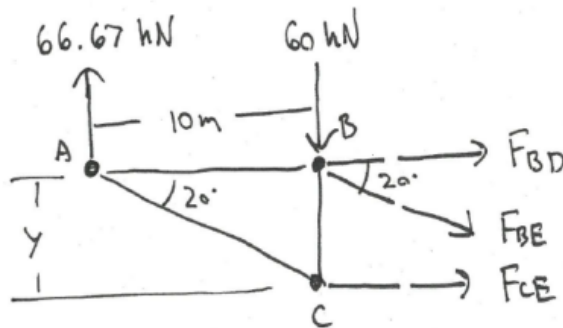
$$\sum M_A = -(60)(10) - (80)(20) + (R_{FY})(30) = 0$$

$$R_{AX} = 0$$

$$R_{FY} = \frac{(60)(10) + (80)(20)}{30} = 73.33 \text{ kN}$$

$$R_{AY} = 60 + 80 - 73.33 = 66.67 \text{ kN}$$

Now cut bridge in half



$$\tan(20^\circ) = \frac{y}{10}$$

$$y = 10 \tan(20^\circ) = 3.64 \text{ m}$$

$$\sum F_x = F_{BD} + \cos(20^\circ) F_{BE} + F_{CE} = 0$$

$$\sum F_y = 66.67 - 60 - \sin(20^\circ) F_{BE} = 0$$

$$\sum M_B = (F_{CE})(3.64) - (66.67)(10) = 0$$

$$F_{BE} = \frac{6.67}{\sin(20^\circ)} = 19.50 \text{ kN}$$

$$F_{CE} = \frac{(66.67)(10)}{3.64} = 183.16 \text{ kN}$$

$$F_{BD} = -\cos(29)(19.50) - 183.16 = -201.48$$

Solution:

$$F_{BD} = 201.48 \text{ kN C}$$

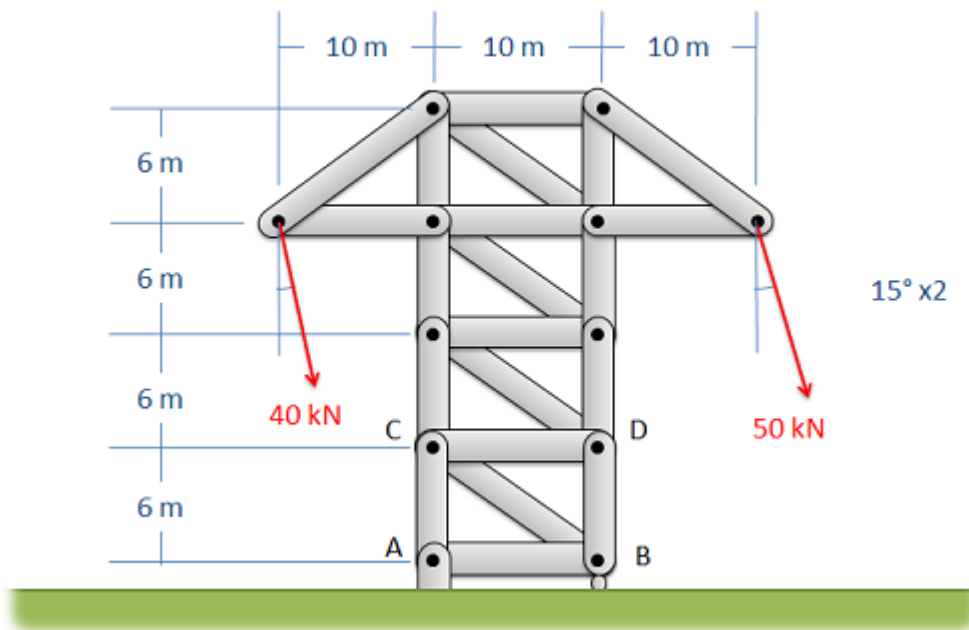
$$F_{CE} = 183.16 \text{ kN T}$$

Source: Engineering Mechanics, Jacob Moore, et al.

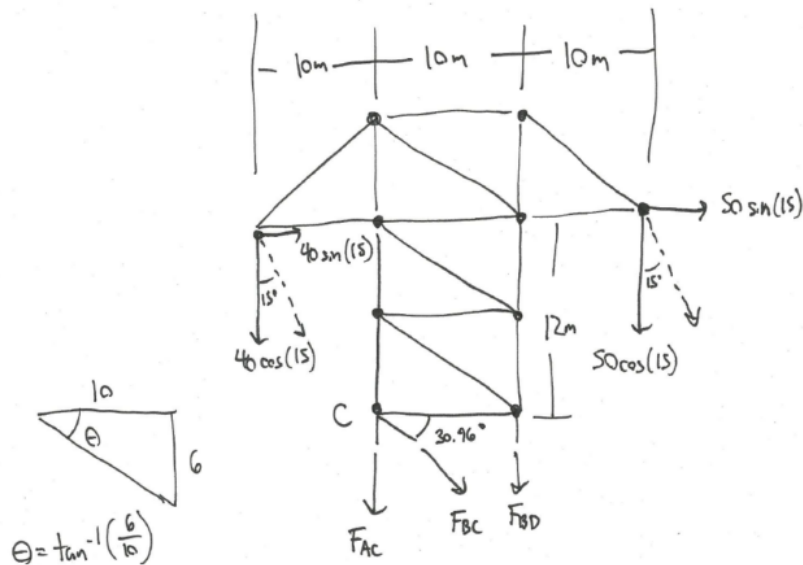
[http://mechanicsmap.psu.edu/websites/5\\_structures/5-5\\_method\\_of\\_sections/pdf/MethodOfSections\\_WorkedExample1.pdf](http://mechanicsmap.psu.edu/websites/5_structures/5-5_method_of_sections/pdf/MethodOfSections_WorkedExample1.pdf)

### Example 2:

Find the forces acting on members AC, BC, and BD of the truss. Be sure to indicate if the forces are tensile or compressive.



If we make a cut in the top section, we don't need to solve for the reaction forces.



$$\sum F_x = 40 \sin(15^\circ) + 50 \sin(15^\circ) + F_{BC} \cos(30.96^\circ) = 0$$

$$\sum F_y = -40 \cos(15^\circ) - 50 \cos(15^\circ) - F_{AC} - F_{BC} \sin(30.96^\circ) - F_{BD} = 0$$

$$\sum M_C = (40 \cos(15^\circ))(10) - (40 \sin(15^\circ))(12) - (50 \cos(15^\circ))(20) + (50 \sin(15^\circ))(12) - F_{BD}(10) = 0$$

$$F_{BC} = \frac{-40 \sin(15^\circ) - 50 \sin(15^\circ)}{\cos(30.96^\circ)} = \boxed{-27.16 \text{ kN}}$$

$$F_{BD} = \frac{40 \cos(15^\circ)(10) - 40 \sin(15^\circ)(12) - 50 \cos(15^\circ)(20) - 50 \sin(15^\circ)(12)}{10}$$

$$\boxed{F_{BD} = -85.91 \text{ kN}}$$

$$F_{AC} = -40 \cos(15) - 50 \cos(15) - \sin(30.46)(-27.16) - (-85.91)$$

$$F_{AC} = 12.95 \text{ kN}$$

Solution:

$$F_{AC} = 12.95 \text{ kN T}$$

$$F_{BC} = 27.16 \text{ kN C}$$

$$F_{BD} = 85.91 \text{ kN C}$$

Source: Engineering Mechanics, Jacob Moore, et al.

[http://mechanicsmap.psu.edu/websites/5\\_structures/5-5\\_method\\_of\\_sections/pdf/MethodOfSections\\_WorkedExample\\_2.pdf](http://mechanicsmap.psu.edu/websites/5_structures/5-5_method_of_sections/pdf/MethodOfSections_WorkedExample_2.pdf)

Even more examples are available at:  
[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.05%3A\\_Internal\\_Forces\\_in\\_Plane\\_Trusses](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.05%3A_Internal_Forces_in_Plane_Trusses)

In summary:



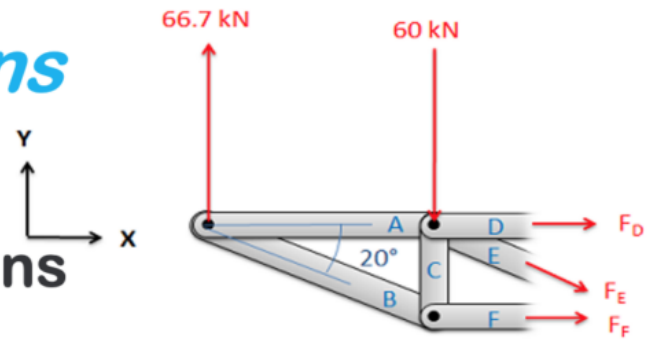
# Method of Sections

## 1) Solve for reactions using equilibrium equations

- *Draw system FBD*
- *Sum forces in  $X, Y$*
- *Sum moments about point with most unknown reaction forces*

## 2) Make a cut & solve for unknowns using equilibrium equations (including moments)

- *Make cut & pick a section (L or R)*
- *Draw a FBD of section*
- *Sum forces in  $X, Y$  and moments of section*
- Use when you only need to know a few members
- Don't always need reaction forces



### Key Takeaways

*Basically:* Method of sections is an analysis technique to find the forces in some members of a truss. It separates the truss into two sections then uses the rigid body equilibrium equations.

*Application:* To calculate the loads on bridges and roofs, especially if you need to know only one or two of the members.

*Looking Ahead:* The next section explores a trick that makes solving faster, especially for method of joints.

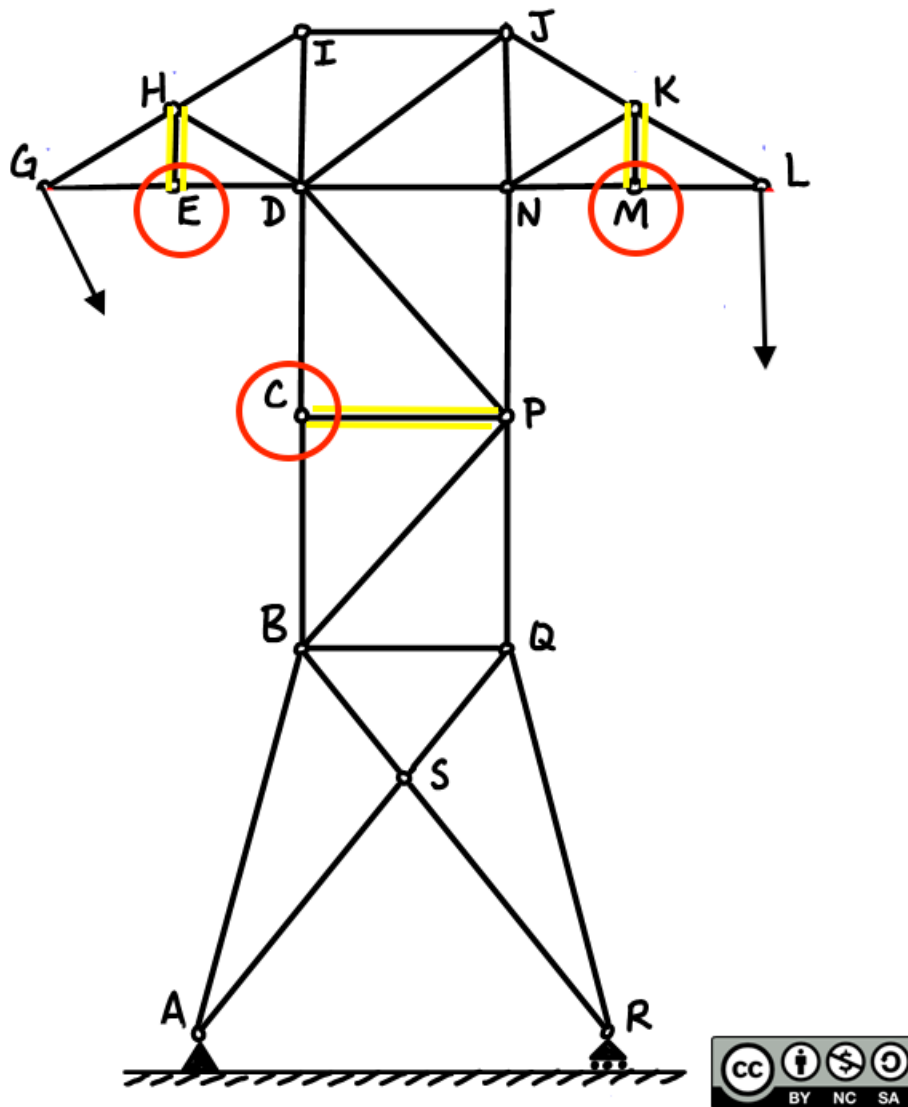


## 5.4 Zero-Force Members

This is a special case that is specially useful for method of joints and method of sections. These special types of members called ***zero-force members*** ensure the truss stays in a particular shape as a rigid body, but carries no load.

Zero-force members are members that you can tell just by inspection that they carry no load. They are important to the structure to ensure it stays in a rigid shape.

Zero-force members can be found considering the equilibrium equations. Look at joint e below. In the y direction, there is only 1 force:  $F_{eh}$ . So if the sum of the forces in the y direction  $\sum F_{eh} = F_{eh} = 0$ , then  $F_{eh} = 0$ . Similarly,  $F_{mk}$  and  $F_{cp}$  are zero-force members (if you look at joint m and c). Note that if you looked at joint k or p, you couldn't tell that  $F_{mk}$  and  $F_{cp}$  are zero-force members.



2013 Michael Swanbom

Adapted from original source:

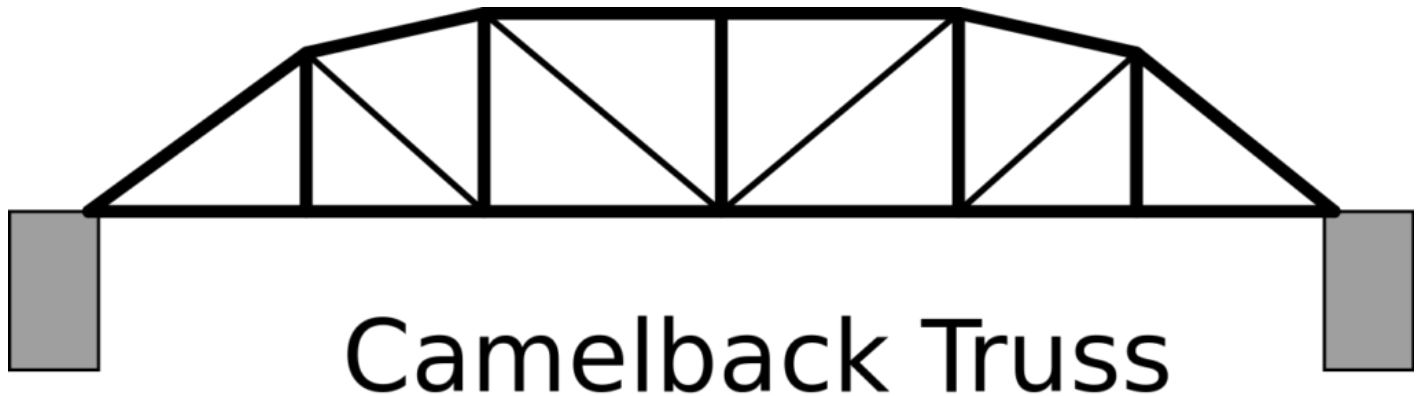
[https://demo.webwork.rochester.edu/webwork2\\_files/tmp/daemon\\_course/images/4cbb3a2-d72c-3d22-bba6-f6856747dafd\\_\\_50b8ddcf-dab2-3209-b817-0ab27426a1d4.png](https://demo.webwork.rochester.edu/webwork2_files/tmp/daemon_course/images/4cbb3a2-d72c-3d22-bba6-f6856747dafd__50b8ddcf-dab2-3209-b817-0ab27426a1d4.png)

There isn't a huge problem if you can't find zero-force members just from inspection, but you might find that certain joints are not able to be solved as easily. (Zero-force members let you have one less unknown).

Also see that L and G have no zero-force members because the externally applied loads balance the members.

Here are some examples to practice on:

### Example 1



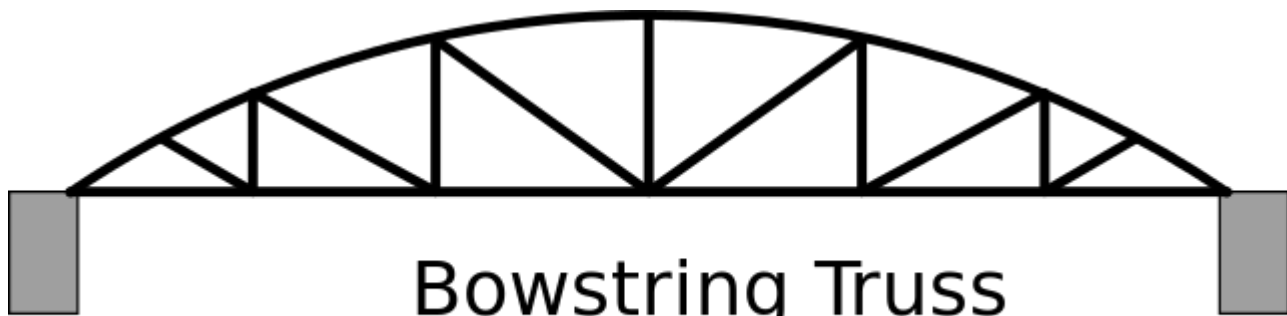
## Camelback Truss

Parker, but with exactly 5 pieces to the top chord

Source: <https://commons.wikimedia.org/wiki/File:Camelback-truss.svg>

(I count 3 zero-force members, assuming there are no loads on the bridge at the joints).

### Example 2



## Bowstring Truss

Top chord is a true arc. Members are often like a Pratt

Source: <https://commons.wikimedia.org/wiki/File:Bowstring-truss.svg>

(I count 1 zero-force member, assuming there are no loads on the bridge at the joints).

## Example 3



Source: <https://pxhere.com/en/photo/995729>

(Looking at only 1 side of the bridge, in theory there are 7 zero-force members, but because there is a load on the deck it is more likely that all of them would be carrying a load).

Admittedly, zero-force members are more theoretical than actual.

### Key Takeaways

*Basically:* Zero-force members are two-force members that do not carry any load but help keep the structure into a certain shape.

*Application:* In trusses.

*Looking ahead:* We will talk about this again in sections 1.3 on vectors and in section 1.4 and 1.5 on dot products and cross products.

# 5.5 Examples

Here are examples from Chapter 5 to help you understand these concepts better. These were taken from the real world and supplied by FSDE students in Summer 2021. If you'd like to submit your own examples, please send them to the author [eosgood@upei.ca](mailto:eosgood@upei.ca).

## Example 5.5.1: Method of Sections - Submitted by Riley Fitzpatrick

### 1. Problem

A flower cart at a local garden center is being pushed with a force of 500 n at joint G. It's back wheels (A) are locked so it is not moving. There is 1 meter of space between each of the four shelves in height and each shelf is four meters long.

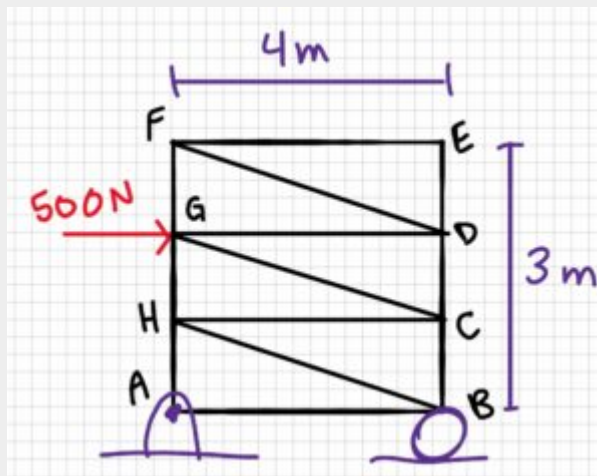
- a) Calculate the reaction forces of the locked wheels and the unlocked wheels.
- b) Calculate the load carried by  $F_{CG}$ , and whether it is in tension or compression

Which method did you use, joints or sections? Which is faster for the style of questions? How would your strategy change if you were calculating the load in each member?



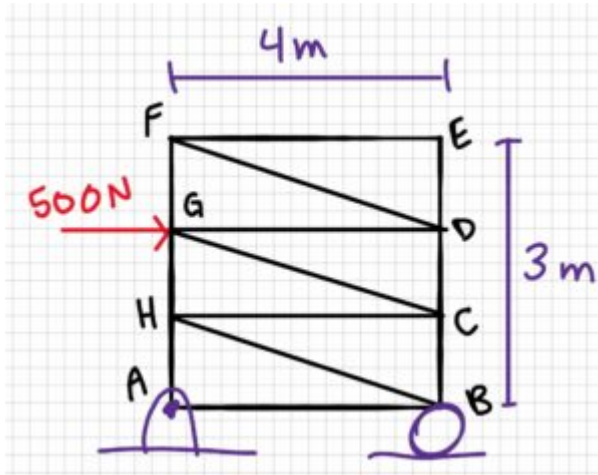


Source: <https://flic.kr/p/txjSpP>

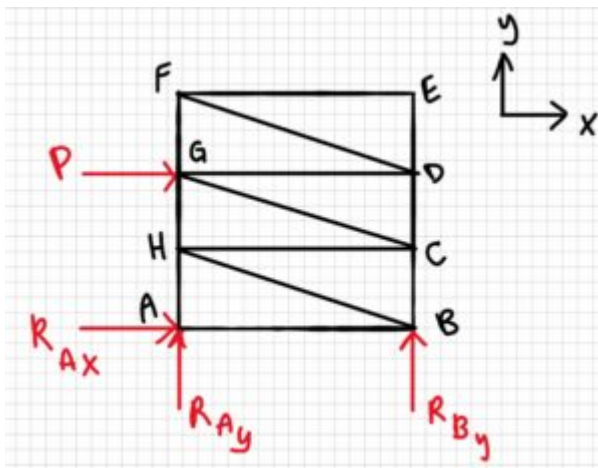


## 2. Draw

Sketch:



Free-body Diagram:



### 3. Knowns and Unknowns

Known:

- $P = 500 \text{ N}$
- Width = 4 m
- Total height = 3 m

Unknowns:  $R_{Ax}$ ,  $R_{Ay}$ ,  $R_{By}$ ,  $F_{CG}$

### 4. Approach

Part a: determine reaction forces using equilibrium equations

Part b: calculate  $F_{CG}$  using method of sections. Make a cut, then solve internal forces using equilibrium equations

### 5. Analysis

Part a:

Solving for  $R_{Ax}$ :

$$\sum F_x = 0 = P - R_{Ax} \implies R_{Ax} = P = 500 \text{ N}$$

Solving for  $R_{By}$ :

$$\sum M_A = 0 = (r_{BA} \cdot R_{By}) - (r_{GA} \cdot P) \quad r_{BA} \cdot R_{By} = r_{GA} \cdot P$$

$$P \cdot R_{By} = \frac{r_{GA} \cdot P}{r_{BA}} \quad R_{By} = \frac{2\text{m} \cdot 500\text{N}}{4\text{m}} \quad R_{By} = 250\text{N}$$

Solving for  $R_{Ay}$ :

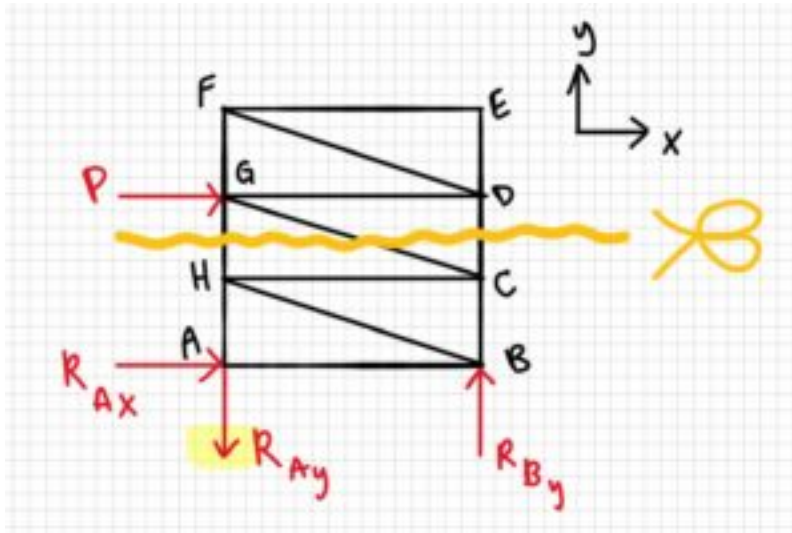
$$\sum F_y = 0 = R_{By} + R_{Ay} \quad R_{Ay} = -R_{By} \quad R_{Ay} = -250\text{N}$$

The answer we got is a negative number. All this means is that the direction of this vector is drawn wrong on our original diagram (in reference to our coordinate frame). This makes sense as  $R_{Ay}$  and  $R_{By}$  are the only external forces in the y direction, so they have to cancel each other for the equilibrium equations to be true. Therefore, one of them should have a negative direction. We will leave this answer as is for now, but the next time we draw the system, we will change the direction of the arrow.

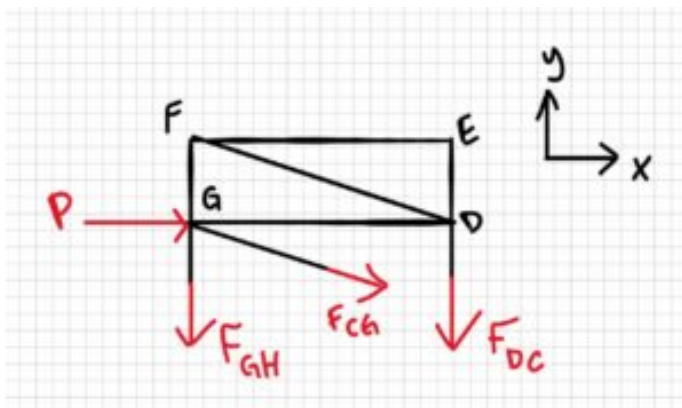
$$\underline{R_{Ax} = 500\text{N}, R_{Ay} = -250\text{N}, R_{By} = 250\text{N}}$$

Part b:

Firstly, we re-draw the diagram, changing the direction of  $R_{Ay}$ . Then, since we are using method of sections, we make a cut so that the member  $F_{CG}$  (the one we want to find) is cut.



Now we re-draw, choosing one of the pieces from the cut. Here the top half is chosen, but you could also choose the bottom half and get the right answer. The only reason the top half was chosen here is because there are less external forces to consider for the top.



Solve for the member we are looking for:

$$\sum F_x = 0 = P + \frac{4}{\sqrt{17}} F_{CG} - \frac{4}{\sqrt{17}} F_{CG} = -P \quad F_{CG} = -P \left( \frac{\sqrt{17}}{4} \right) = -500 \text{ N} \left( \frac{\sqrt{17}}{4} \right) = -515.388 \text{ N}$$

Again, the number we get is negative. The way we drew  $F_{CG}$  originally was as if the member was in tension. The negative number just means that it is actually compression, not tension.

$$\underline{F_{CG} = 515 \text{ N (Compression)}}$$

Part c:

For part b I used the method of sections, as it would be the fastest method. The method of joints would require the lower joints to be solved first which would be a much slower process, whereas with this method a simple cut can be made and the member's load can be quickly solved using equilibrium equations. Had the question asked for all member loads to be solved however, the method of joints would have been the faster approach.

## 6. Review

Part a:

$R_{Ax}$  is equal and opposite to  $P$  so we know it is correct, and the value of  $R_{By}$  should also be correct as its moment about A ( $250 \times 4 \text{ m} = 1000 \text{ Nm}$ ), is equal and opposite to the moment of  $P$  about A, ( $500 \text{ N} \times 2 \text{ m} = 1000 \text{ Nm}$ ). as  $R_{Ay}$  is equal and opposite to  $R_{By}$  it is also correct.

Part b:

The x component of the calculated value of  $F_{CG}$  is equal in magnitude to  $P$  (see equation below), and it is the only cut member acting in the x direction. Therefore, it must be correct.

$$\frac{4}{\sqrt{17}} (515 \text{ N}) = 500 \text{ N}$$

Part c:

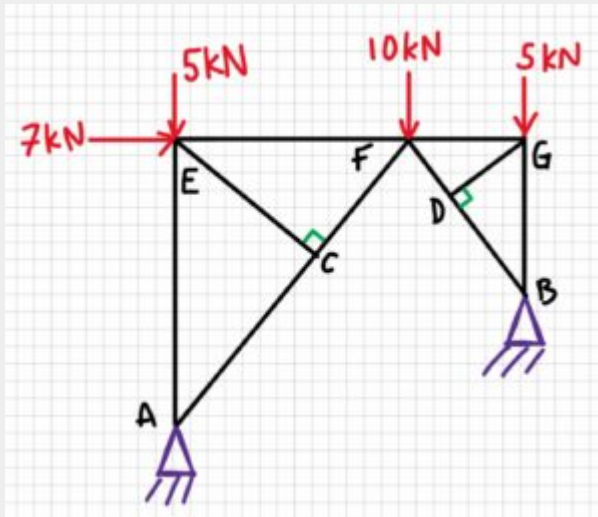
The method of sections allows you to solve a very specific area of the systems internal forces (the members that are cut), whereas the method of joints usually requires you to solve most if not all of the internal forces of the system. Therefore, the method of sections is the most efficient for finding the internal forces of specific parts of the system, whereas the method of joints is more efficient for solving the whole system.

## Example 5.5.2: Zero-Force Members, submitted by Michael Oppong-Ampomah

### 1. Problem

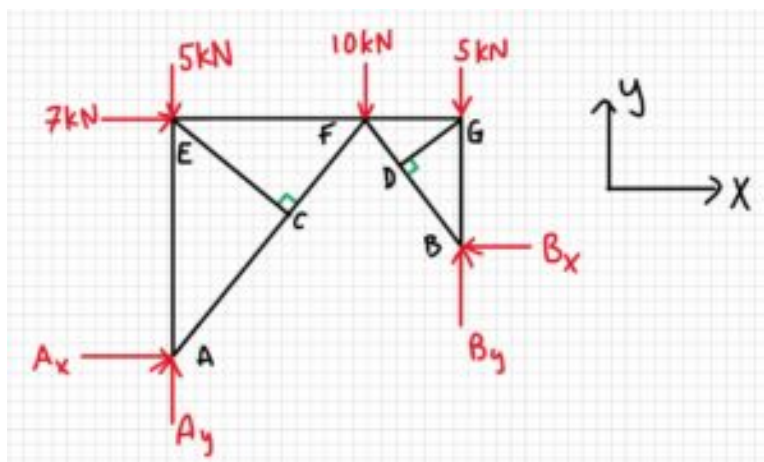
A bridge with uneven ground has been built as shown below. Force is applied at three points on the top of the bridge.

- Find any zero-force members
- What purpose do these members serve?



## 2. Draw

Free-body diagram:



## 3. Knowns and Unknowns

Unknown: which members are zero-force

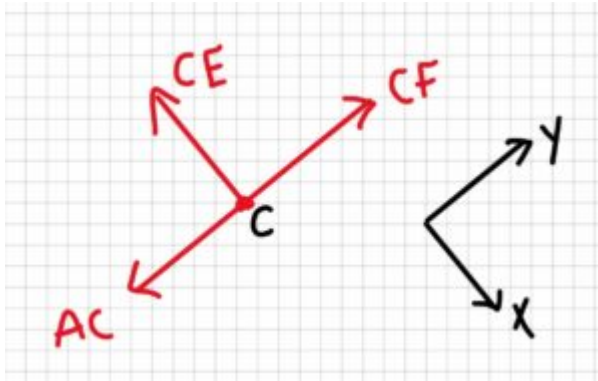
## 4. Approach

Look at each joint and determine how many forces are in each direction. If there is only one force in a direction, that member is zero-force.

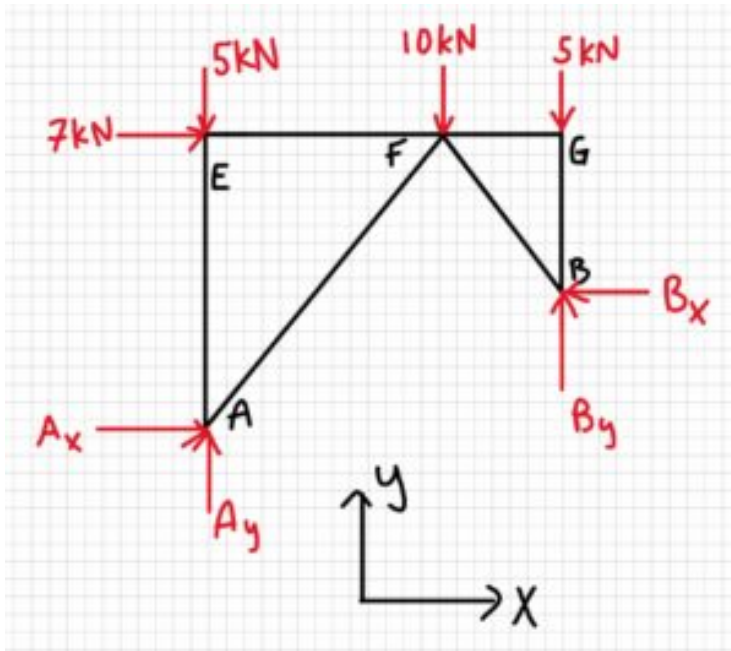
### 5. Analysis

Part a:

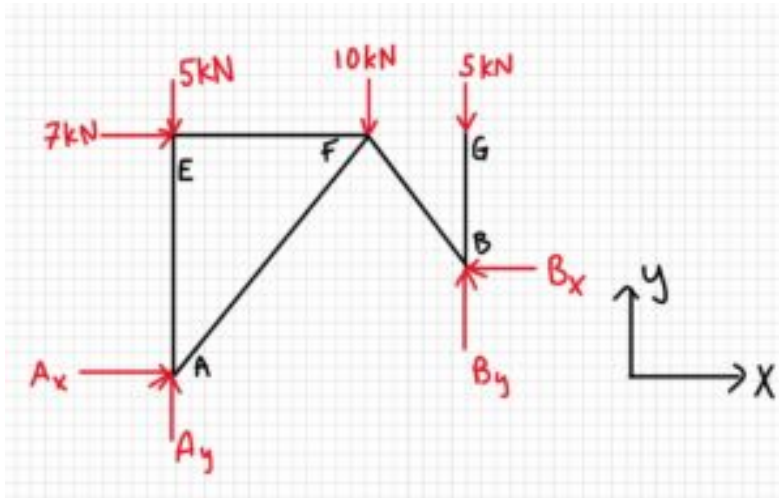
Let's start with joint C. If we think of the forces acting in the x and y directions as shown below by the coordinate frame, we see that there are two forces acting in the y direction, and only one in the x direction. Therefore, assuming the joint is in static equilibrium, member CE is zero-force.



If we do the same type of analysis for the other joints and remove the zero-force members, the structure now looks like this:



After one more analysis of the joints, we find one more zero-force member, as shown below.



Answer: CE, DG, and FG are zero-force members.

Part b:

Zero-force members exist to provide stability to the truss, to keep the shape rigid.

### 6. Review

Although the new truss (without zero-force members) looks strange, there are no joints where there's only one force in one direction, therefore there are no more zero-force members.



# CHAPTER 6: INTERNAL FORCES

In the last chapter we looked at the normal (axial) force running through beams joined into trusses by analyzing either the joints or a whole section of the truss.

In this chapter, we look at what happens *along* a *single* beam. We will look at three types of internal forces and moments. Note that when we say 'internal forces', we really mean 'internal forces and moments'. Inside a beam, we will calculate the normal and shear forces as well as the bending moment at any point in the beam.

For this chapter: the shear force and bending moment change throughout the beam because additional transverse forces are applied. However, the normal force usually stays the same, because it's uncommon to have applied axial forces along the beam.

Here are the sections in this Chapter:

- 6.1 Types of Internal Forces - shear force, normal force and bending moment
- 6.2 Shear/Moment Diagrams - graphing the shear force and bending moment
- 6.3 Examples - Examples from your peers

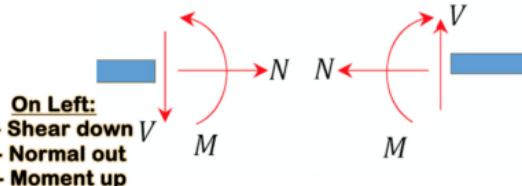
Here are the important equations for this chapter:

## Ch 6 Equations

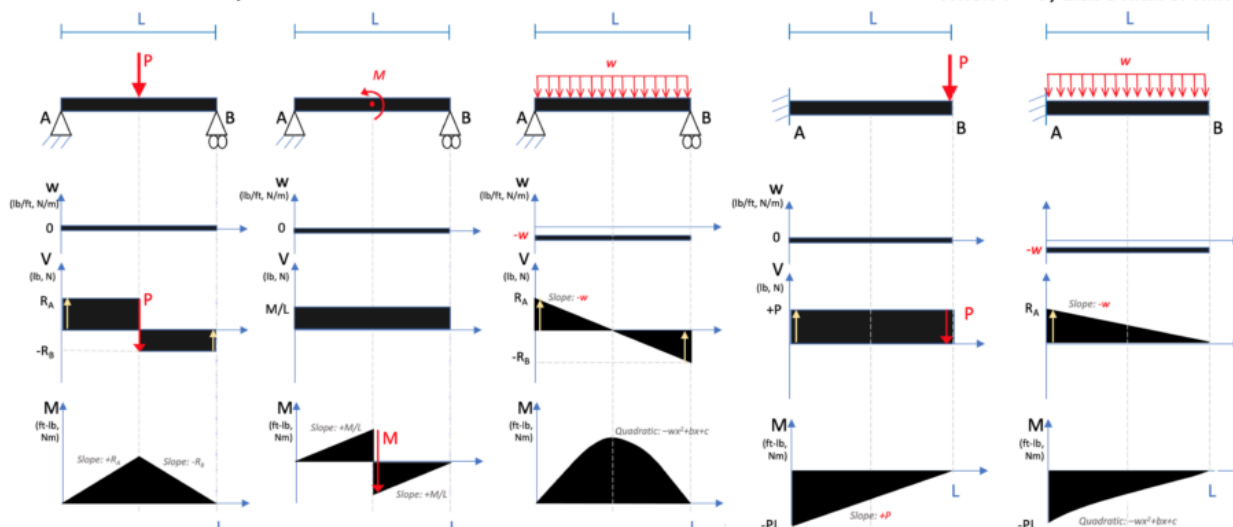
1. Calculate Reactions
2. Identify type of loads/inflexion points
3. Estimate shape of Shear & Moment diagrams
4. Calculate shear & moment for each inflection point

### Positive Sign Convention

<https://eng.libretexts.org/Bookshelves/Engineering%3DStatics/Book%3DStatics%3DBookshelves/Engineering%3DStatics/Chapter%3D6%3DInternal%3DForces%3Dand%3DMoments>



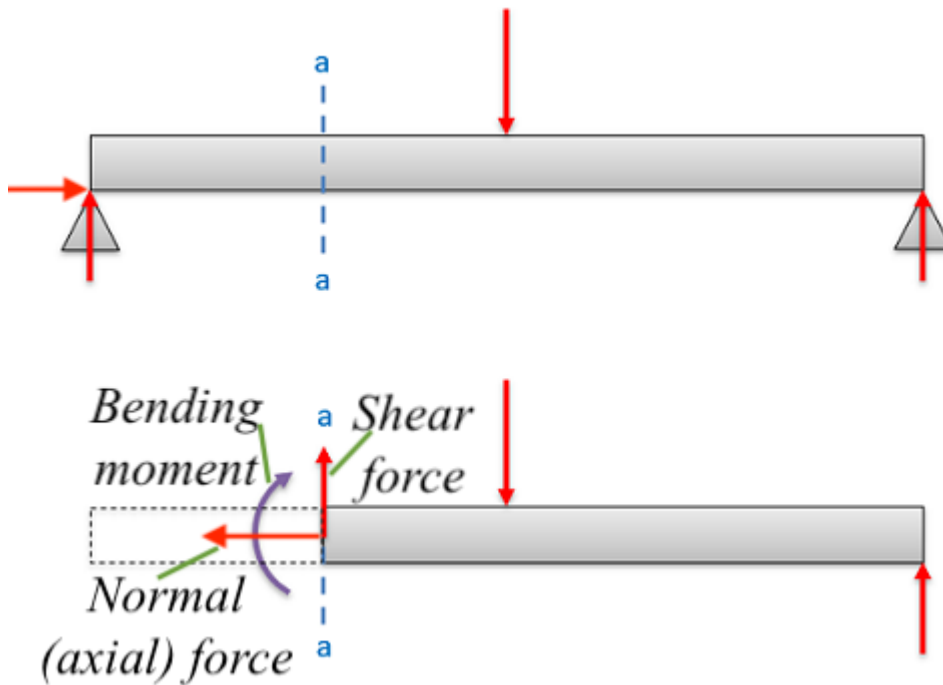
- If no applied load/moment at ends:  
Cantilever: Nonzero V & M at start, 0 at end  
Simply supported: Start and end with reaction forces for V, Start and end at zero for M
- Where are the jumps?  
  - In V, Forces jump up or down at applied loads matching direction
  - In M, moments jump at applied moments
- +V means increasing M
- -V means decreasing M
- When V = 0, that's max or min M





# 6.1 Types of Internal Forces

When you make a cut in an object, similar to a fixed reaction, we describe what is happening at that point using one horizontal force (called normal force), one vertical force (called shear force), and a bending moment.



Adapted from source: *Engineering Mechanics, Jacob Moore, et al.*  
[http://mechanicsmap.psu.edu/websites/6\\_internal\\_forces/6-2\\_internal\\_forces\\_equilibrium/internal\\_forces\\_equilibrium.html](http://mechanicsmap.psu.edu/websites/6_internal_forces/6-2_internal_forces_equilibrium/internal_forces_equilibrium.html)

## 6.1.1 Types of Internal Forces

There are 3 types of internal forces (& moments):

- normal force (N) – the horizontal force we calculated in trusses in the last chapter
- shear force (V) – the vertical force that changes based on the applied loads
- bending moment (M) – changes based on the applied loads and applied moments

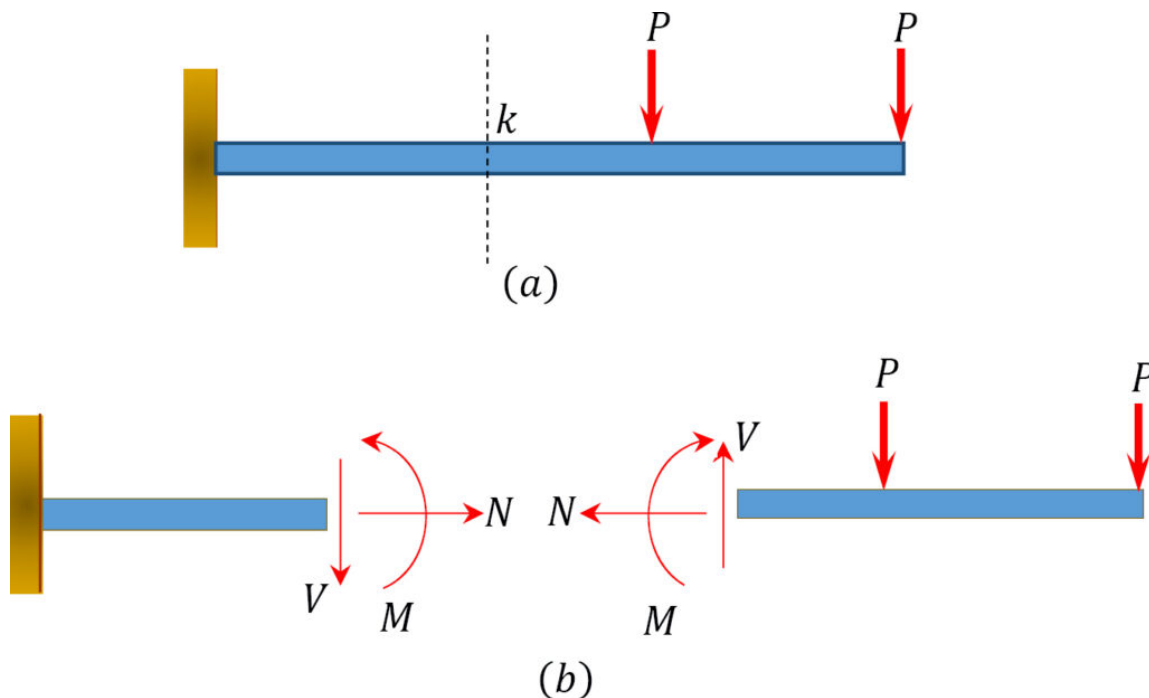
Normal force is represented by 'N'. Shear force, the vertical force is represented with 'V'. Bending moment is 'M'. Normal and shear have units of N or lb and bending moment has units of Nm or ft-lb. The following table summarizes information on internal forces (and moments).

<b>Force/Moment</b>	<b>Abbreviation</b>	<b>Unit</b>	<b>Direction for a horizontal beam</b>
Normal Force	N	N or lb	horizontal
Shear Force	V	N or lb	vertical
Moment	M	Nm or ft-lb	rotation

Note that for a vertical column, the normal force would be vertical. For this reason, the normal force is often called 'axial' as in: along the axis. The shear force for a column would be horizontal and is sometimes called 'transverse'.

This is for a 2d analysis of the beam assuming there is negligible loading in the third dimension.

When a beam or frame is subjected to transverse loadings, the three possible internal forces that are developed are the normal or axial force, the shearing force, and the bending moment, as shown in section  $k$  of the cantilever of the figure below. To predict the behavior of structures, the magnitudes of these forces must be known. In this chapter, the student will learn how to determine the magnitude of the shearing force and bending moment at any section of a beam or frame and how to present the computed values in a graphical form, which is referred to as the "shearing force" and the "bending moment diagrams." Bending moment and shearing force diagrams aid immeasurably during design, as they show the maximum bending moments and shearing forces needed for sizing structural members.



## Normal Force

The normal force at any section of a structure is defined as the algebraic sum of the axial forces acting on either side of the section.

## Shearing Force

The shearing force (SF) is defined as the algebraic sum of all the transverse forces acting on either side of the section of a beam or a frame. The phrase “on either side” is important, as it implies that at any particular instance the shearing force can be obtained by summing up the transverse forces on the left side of the section or on the right side of the section.

## Bending Moment

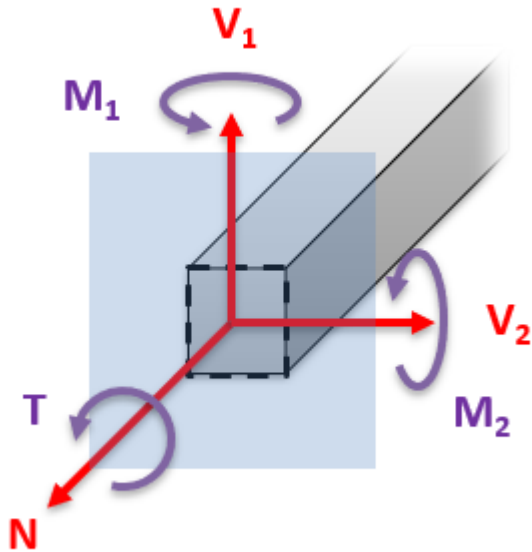
The bending moment (BM) is defined as the algebraic sum of all the forces' moments acting on either side of the section of a beam or a frame.

Source: Internal Forces in Beams and Frames, Libretexts.

[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames)

In 3 dimensions, there are:

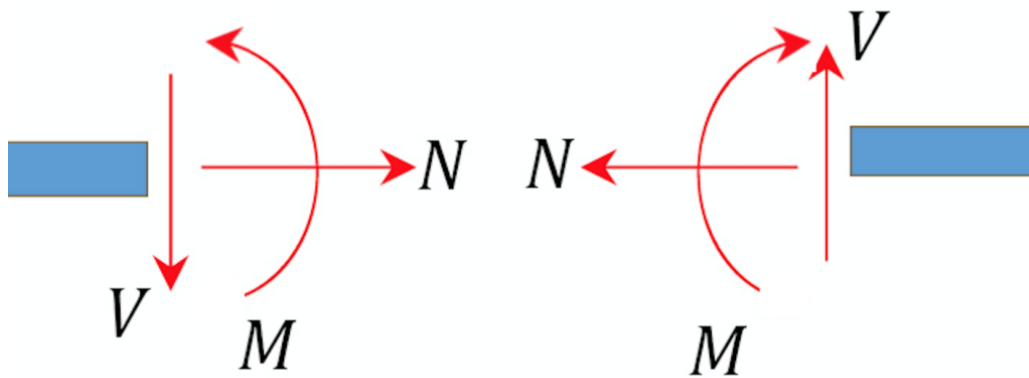
- 1 normal force ( $N$ )
- 2 shear forces ( $V_1$  &  $V_2$ ), and
- 3 bending moments ( $M_1$ ,  $M_2$ , &  $T$  – torsion).



Source: Engineering Mechanics, Jacob Moore, et al.  
[http://mechanicsmap.psu.edu/websites/6\\_internal\\_forces/6-2\\_internal\\_forces\\_equilibrium/internal\\_forces\\_equilibrium.html](http://mechanicsmap.psu.edu/websites/6_internal_forces/6-2_internal_forces_equilibrium/internal_forces_equilibrium.html)

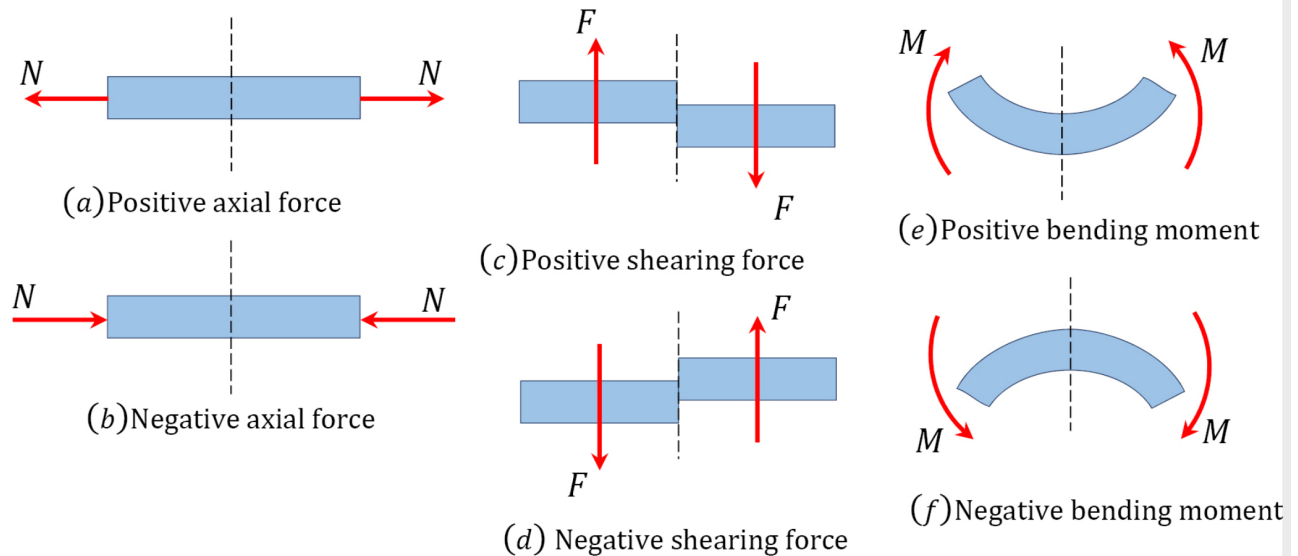
## 6.1.2 Sign Convention

So that there is a standard within the industry, a sign convention is necessary so we agree on what is positive and what is negative. On the right for shear - up is positive. Notice that both of the following figures show the identical sign convention.



Positive sign convention adapted from source:  
[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames)

When you look at the beam as a whole (in the figure below), positive shear is right side down. When you cut into beam, for it to be in static equilibrium, the positive shear must then be up on the right to be equal and opposite of the overall motion.



## Axial (Normal) Force

An axial force is regarded as positive if it tends to tier the member at the section under consideration. Such a force is regarded as tensile, while the member is said to be subjected to axial tension. On the other hand, an axial force is considered negative if it tends to crush the member at the section being considered. Such force is regarded as compressive, while the member is said to be in axial compression.

## Shear Force

A shear force that tends to move the left of the section upward or the right side of the section downward will be regarded as positive. Similarly, a shear force that has the tendency to move the left side of the section downward or the right side upward will be considered a negative shear force.

## Bending Moment

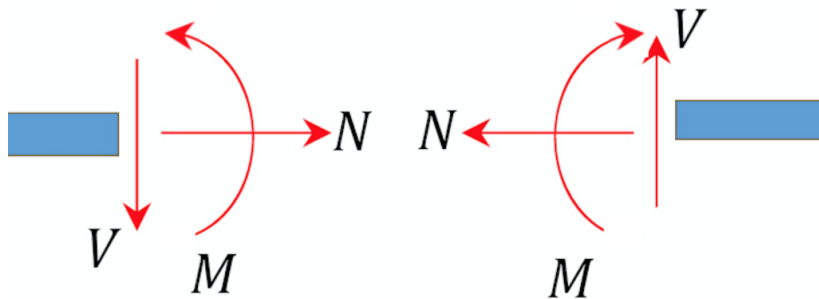
A bending moment is considered positive if it tends to cause concavity upward (sagging). If the bending moment tends to cause concavity downward (hogging), it will be considered a negative bending moment.

Source: Internal Forces in Beams and Frames, Libretexts.

[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames)

## 6.1.3 Calculating the Internal Forces

To solve the internal forces at a certain point along the beam,

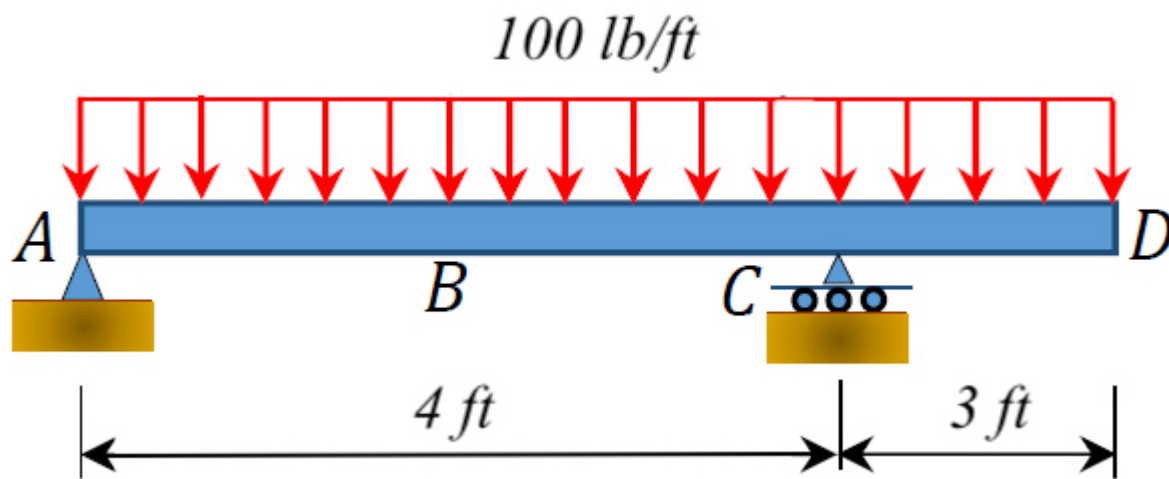


Positive sign convention adapted from

[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames)

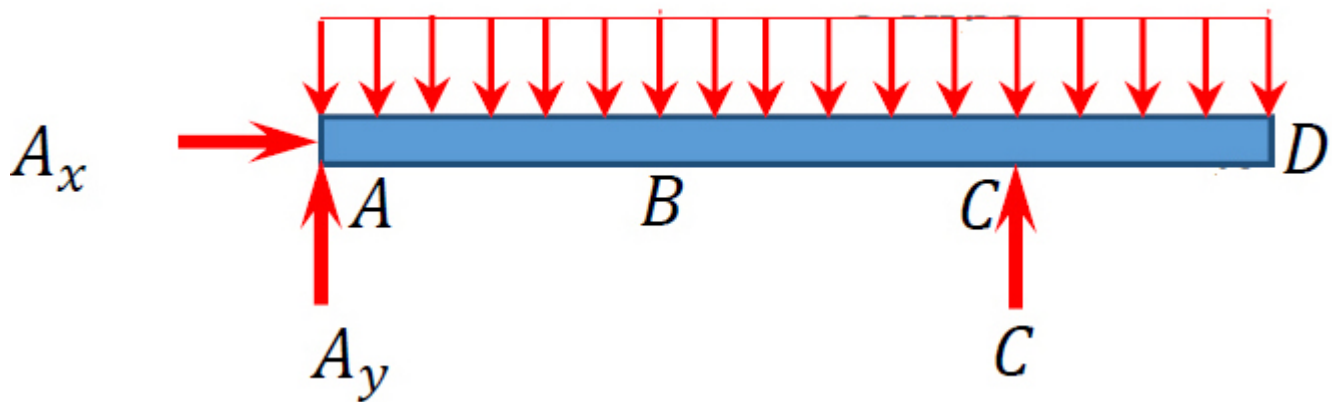
1. Find the external & reaction forces
2. Make a cut.
3. In a FBD of one side of the cut, add the internal forces (and moments) using the positive sign convention.
4. Use the equilibrium equations to solve for the unknown internal forces and moments.

Example: For the following distributed load, a) what are reaction forces? b) what are the internal forces at the midpoint b) between reaction forces?



Adapted from: Source: Engineering Mechanics, Jacob Moore, et al.  
[http://mechanicsmap.psu.edu/websites/6\\_internal\\_forces/6-3\\_axial\\_torque\\_diagrams/axial\\_torque\\_diagrams.html](http://mechanicsmap.psu.edu/websites/6_internal_forces/6-3_axial_torque_diagrams/axial_torque_diagrams.html)

1. Solve external forces:



Adapted from: Source: Engineering Mechanics, Jacob Moore, et al.  
[http://mechanicsmap.psu.edu/websites/6\\_internal\\_forces/6-3\\_axial\\_torque\\_diagrams/axial\\_torque\\_diagrams.html](http://mechanicsmap.psu.edu/websites/6_internal_forces/6-3_axial_torque_diagrams/axial_torque_diagrams.html)

$$\sum F_x = A_x = 0$$

$$\sum F_y = A_y + C - \omega L = 0$$

$$\sum M_A = -(\omega L)\left(\frac{L}{2}\right) + d_{AC} C = 0$$

$$C = \left(\frac{\omega L^2}{2d_{AC}}\right) = \frac{(100 \text{ lb/ft})(7\text{ft})^2}{2 * (4\text{ft})} = 612.5 \text{ lb (} +j \text{ direction)}$$

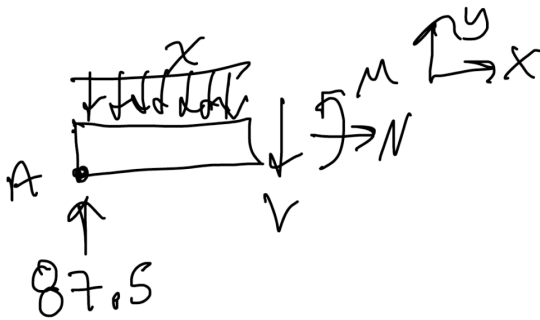
$$A_y = \omega L - C = (100 \text{ lb/ft})(7\text{ft}) - 612.5 \text{ lb} = 87.5 \text{ lb (} +j \text{ direction)}$$

$$\underline{A_x = 0 \quad A_y = 87.5 \text{ lb (} +j \text{)} \quad C = 612.5 \text{ lb (} +j \text{)}}$$



2. Make a cut at B.

3. In a FBD of one side of the cut, add the internal forces (and moments) using the positive sign convention.



4. Use the equilibrium equations to solve for the unknown internal forces and moments.

For just this portion, the force from intensity is:  $F_w = (100 \text{ lb/ft}) * (2 \text{ ft}) = 200 \text{ lb}$  and acts 1 ft from the left, so the moment due to intensity is:  $M_w = w * 2 \text{ ft} * 1 \text{ ft} = F_w * 1 \text{ ft} = (100 \text{ lb/ft}) * (2 \text{ ft}) * (1 \text{ ft}) = 200 \text{ ft-lb}$

$$\sum F_y = 87.5 \text{ lb} - 200 \text{ lb} - V = 0 \quad V = -112.5 \text{ lb} \quad \text{(- indicates going up not down)}$$

$$\sum M_A = - (w * 2 \text{ ft}) * (1 \text{ ft}) - V * (2 \text{ ft}) + M = 0 \quad M = (100 \frac{\text{lb}}{\text{ft}}) * 2 \text{ ft}^2 + (-112.5 \text{ lb}) * (2 \text{ ft}) \quad M = 200 \text{ ft} \cdot \text{lb} - 225 \text{ ft} \cdot \text{lb} \quad M = -25 \text{ ft} \cdot \text{lb} \quad \text{(- indicates going reverse direction)}$$

$$\underline{N = 0 \quad V = -112.5 \text{ lb} \quad (+j) \quad M = -25 \text{ ft} \cdot \text{lb} \quad \text{(clockwise)}}$$

Key Takeaways



*Basically:* The internal forces (and moments) for a 2d beam are: shear, normal, and bending moment. There is a positive sign convention to use when making a cut along a beam to determine the forces inside: on the left: shear down, normal out, moment up.

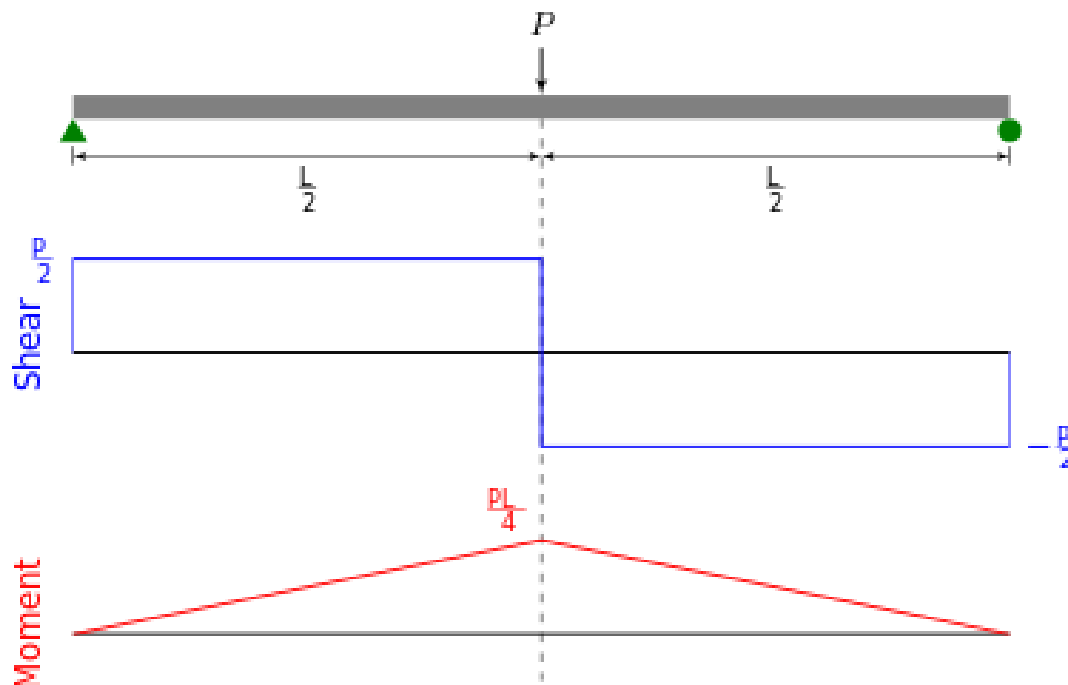
*Application:* A bridge that has different loads applied (from cars, trucks, lampposts, etc). Use this method to calculate the internal loads at a particular point of interest.

*Looking Ahead:* In the next section, we'll look at how to calculate the internal force across the whole beam, and display the results graphically.

## 6.2 Shear/Moment Diagrams

### 6.2.1 What are Shear/Moment Diagrams?

Shear/Moment diagrams are graphical representations of the internal shear force and bending moment along the whole beam.



Source (image): By XFEM Skier - Own work, CC BY-SA 3.0,  
<https://commons.wikimedia.org/w/index.php?curid=29178249>

#### Shearing Force Diagram

This is a graphical representation of the variation of the shearing force on a portion or the entire length of a beam or frame. As a convention, the shearing force diagram can be drawn above or below the  $x$ -centroidal axis of the structure, but it must be indicated if it is a positive or negative shear force.

## Bending Moment Diagram

This is a graphical representation of the variation of the bending moment on a segment or the entire length of a beam or frame. As a convention, the positive bending moments are drawn above the x-centroidal axis of the structure, while the negative bending moments are drawn below the axis.

Below is a simple example of what shear and moment diagrams look like, afterwards, the relation between the load on the beam and the diagrams will be discussed.

Source: Internal Forces in Beams and Frames, LibreTexts.

[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames)

## 6.2.2 Distributed Loads & Shear/Moment Diagrams

There is a relationship between distributed loads and shear/moment diagrams. Simply put:

$$\frac{dM}{dx} = V(x)$$

$$\frac{dV}{dx} = -w(x)$$

$$\frac{d^2M}{dx^2} = -w(x)$$

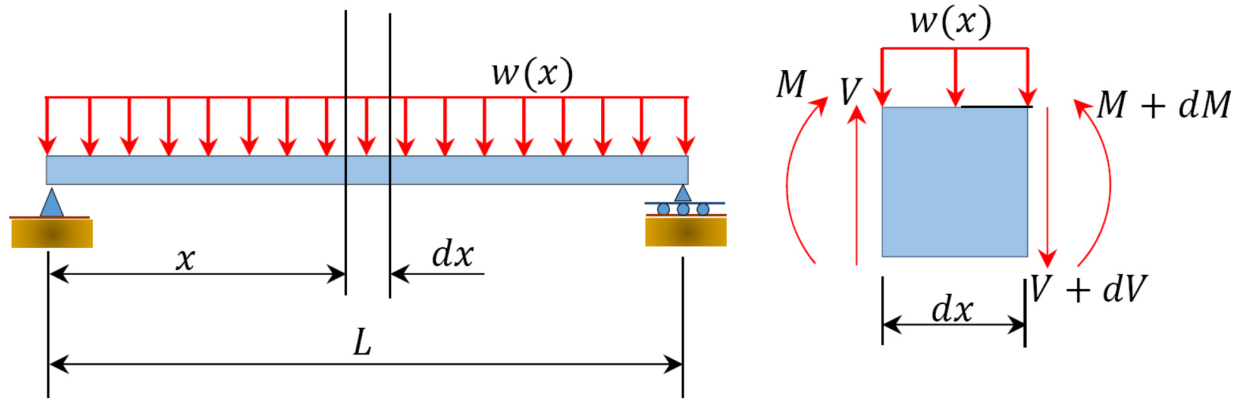
Or:

$$\Delta M = \int V(x) dx$$

$$\Delta V = \int w(x) dx$$

So, if there is a constant distributed load, then the slope of shear will be linear and the slope of the moment will be parabolic. If distributed load is 0, then the shear will be constant and the slope of the moment will be linear (as shown in example 1 in the next section).

For the derivation of the relations among  $w$ ,  $V$ , and  $M$ , consider a simply supported beam subjected to a uniformly distributed load throughout its length, as shown in the figure below. Let the shear force and bending moment at a section located at a distance of  $x$  from the left support be  $V$  and  $M$ , respectively, and at a section  $x + dx$  be  $V + dV$  and  $M + dM$ , respectively. The total load acting through the center of the infinitesimal length is  $w dx$ .



To compute the bending moment at section  $x + dx$ , use the following:

$$M_{x+dx} = M + Vdx - w dx \cdot \frac{dx}{2} \approx M + Vdx$$
 (neglecting the small second order term  $w dx^2/2$ )

$$M + dM = M + Vdx$$

or

$$\frac{dM}{dx} = V(x) \quad \text{(Equation 6.1)}$$

Equation 6.1 implies that the first derivative of the bending moment with respect to the distance is equal to the shearing force. The equation also suggests that the slope of the moment diagram at a particular point is equal to the shear force at that same point. Equation 6.1 suggests the following expression:

$$\Delta M = \int V(x) dx \quad \text{(Equation 6.2)}$$

Equation 6.2 states that the change in moment equals the area under the shear diagram. Similarly, the shearing force at section  $x + dx$  is as follows:

$$V_{x+dx} = V - w dx \approx V - w dx$$

or

$$\frac{dV}{dx} = -w(x) \quad \text{(Equation 6.3)}$$

Equation 6.3 implies that the first derivative of the shearing force with respect to the distance is equal to the intensity of the distributed load. Equation 6.3 suggests the following expression:

$$\Delta V = -\int w(x) dx \quad \text{(Equation 6.4)}$$

Equation 6.4 states that the change in the shear force is equal to the area under the load diagram. Equation 6.1 and 6.3 suggest the following:

$$\frac{d^2M}{dx^2} = -w(x) \quad (\text{Equation 6.5})$$

Equation 6.5 implies that the second derivative of the bending moment with respect to the distance is equal to the intensity of the distributed load.

Source: Internal Forces in Beams and Frames, LibreTexts.

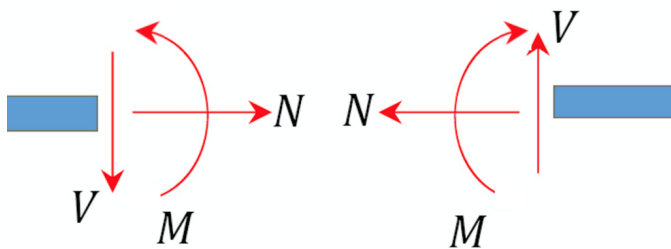
[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames)

## 6.2.3 Producing a Shear/Moment Diagram

There are many methods you can use to solve a shear/moment diagram. First, you can find the equation for each portion and integrate using the above equations.

Second, you could use the method shown in the previous section to calculate the internal forces at important points (where loads are applied, the start and end of distributed loads, at reaction points). Plot these points on the V and M plots at the x locations, then connect the dots using the appropriate shape slope (more on this at the bottom of this page).

Third, you can find the equations by using the equilibrium equations (so there's no integration/differentiation).



*Adapted from original source*

[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames)

Draw a FBD of the structure

2. Calculate the reactions using the equilibrium equations (may not need to do this if choosing a cantilever beam and using the free side for the FBD).

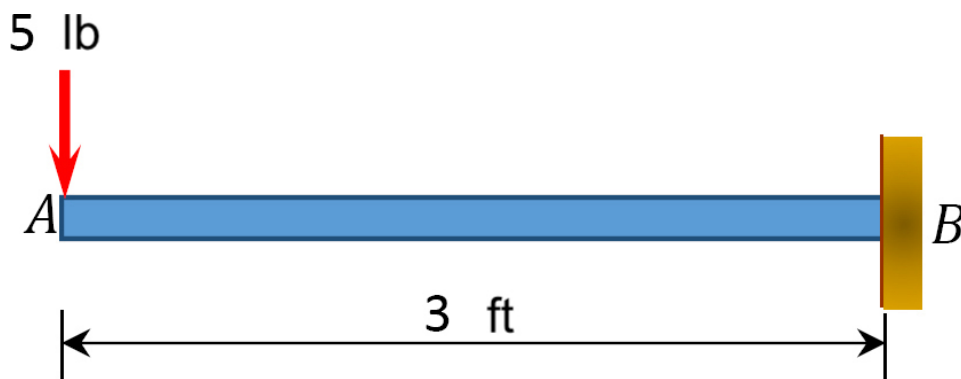
3. Make a cut and add internal forces  $N$ ,  $V$  and  $M$  using the positive sign convention. Depending on the number of loads, you may need multiple cuts. Recall the positive convention:
4. For shear, find an equation (expression) of the shear that is  $x$  distance from the origin (often the reaction) for each cut.
5. For moment, find an equation (expression) of the shear that is  $x$  distance from the origin (often the reaction) for each cut.
6. Plot these equations on a plot on top of each other.

The rest of this section will use this method.

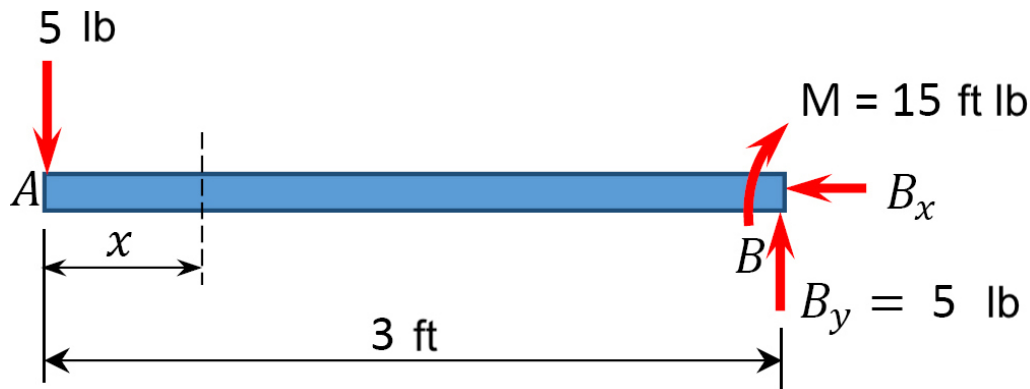
## Example 1

(adapted from [https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames))

Draw the shear force and bending moment diagrams for the cantilever beam supporting a concentrated load of 5 lb at the free end 3 ft from the wall.



1. Draw a FBD of the structure



2. Calculate the reactions using the equilibrium equations (may not need to do this if choosing a cantilever beam and using the free side for the FBD).

First, compute the reactions at the support. Since the support at  $B$  is fixed, there will be three reactions at that support, namely  $B_y$ ,  $B_x$ , and  $M_B$ . Applying the conditions of equilibrium suggests the following:

$$\sum F_x = 0: \quad \underline{B_x = 0}$$

$$\sum F_y = 0: \quad -5 \text{ lb} + B_y = 0$$

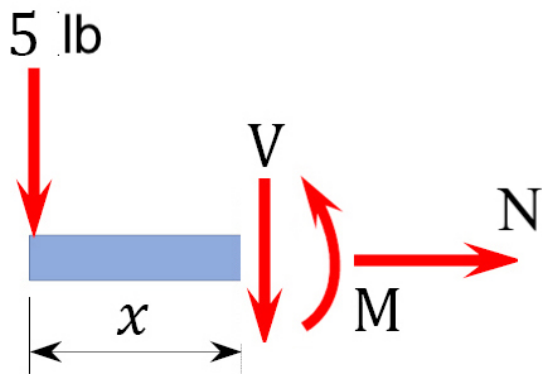
$$\quad \quad \underline{B_y = 5 \text{ lb}}$$

$$\sum M_B = 0: \quad (5 \text{ lb})(3 \text{ ft}) - M = 0$$

$$\quad \quad \underline{M = 15 \text{ ft} \cdot \text{lb}}$$

3. Make a cut and add internal forces  $N$ ,  $V$  and  $M$  using the positive sign convention. Depending on the number of loads, you may need multiple cuts

Only 1 cut needed because only 1 load is added at the end. (If it were in the middle there would be 2 sections to consider). The value  $x$  could be 0 to 3 ft.



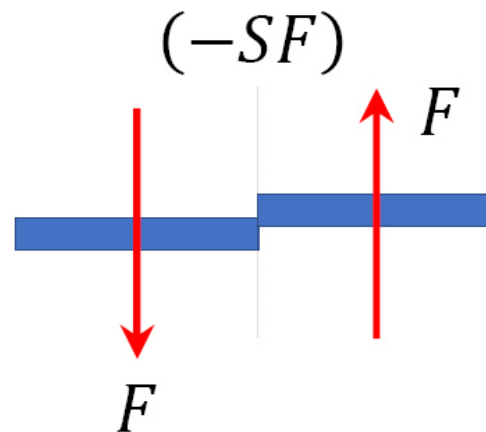
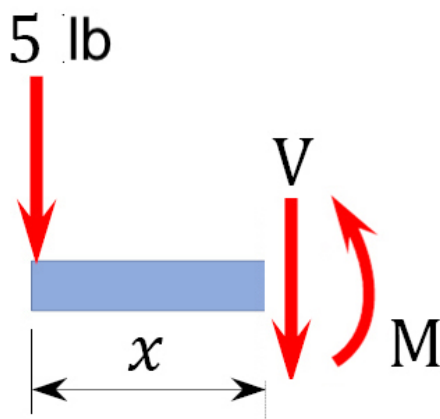
4. For shear, find an equation (expression) of the shear that is  $x$  distance from the origin (often the reaction) for each cut.

$x$  is the distance from the free end of the cantilever beam to the cut. The shearing force at that section is due to the applied load. Using the equilibrium equations,

$$\sum F_y = -5 \text{ lb} - V = 0 \quad \quad \underline{V = -5 \text{ lb}} \quad \text{(- indicates } V \text{ acts in opposite direction)}$$

The constant number for shear means that it doesn't change or vary by  $x$ . (If there were a distributed load,  $x$  would be part of the equation).

The negative sign indicates the shear actually goes the opposite direction. (This is due to the fact that the sign convention for a shearing force states that a downward transverse force on the left of the section under consideration will cause a negative shearing force on that section.)



5. For the moment, find an equation (expression) of the shear that is  $x$  distance from the origin (often the reaction) for each cut.

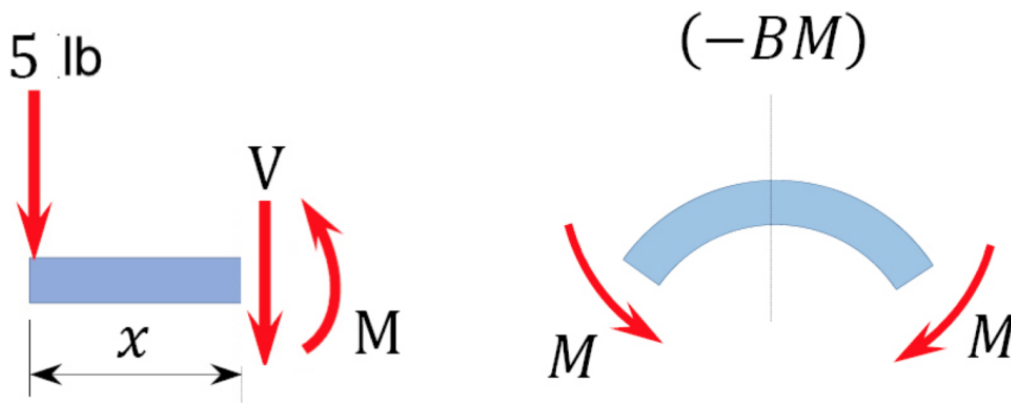
Here,  $x$  is measured from the left. Using sum of the moments equations, find an expression for  $M$ . You could choose to sum the moments about the end point where the load is applied, or you could do it at the moving point  $x$ . Both take the same effort for this problem, so let's choose the left hand side where the  $5 \text{ lb}$  are being applied.

$$\sum M_L = -Vx - M = 0$$

$$\quad \quad \quad M = + Vx = (-5 \text{ lb}) * x$$

$$\quad \quad \quad \underline{M = -(5 \text{ lb})x} \quad \text{(the negative sign indicates the arrow goes the other direction.)}$$



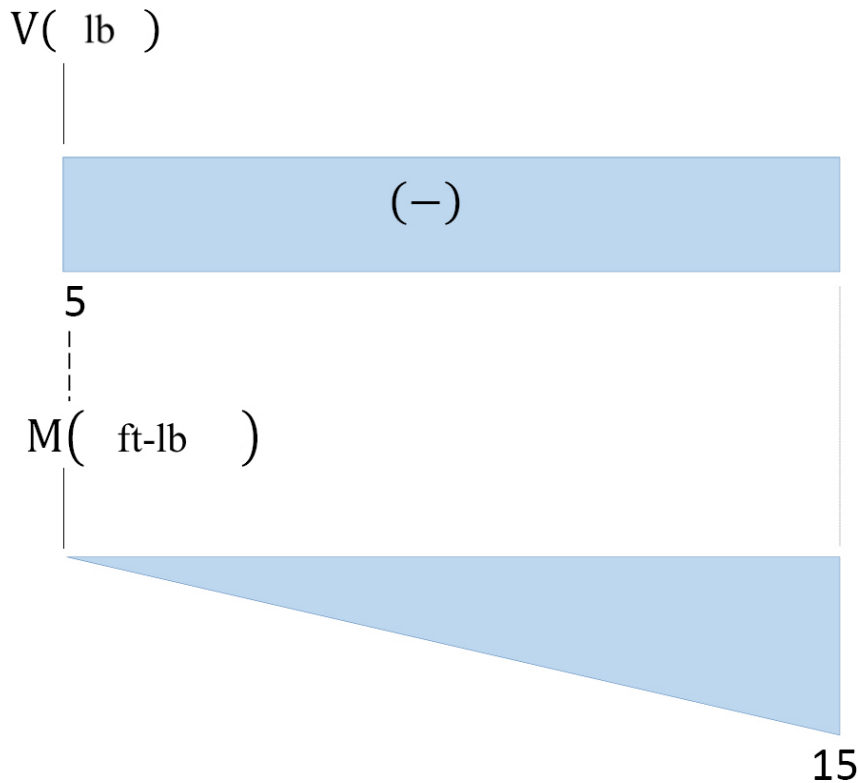


The obtained expression is valid for the entire beam (the region  $0 < x < 3$  ft). The negative sign indicates a negative moment, which was established from the sign convention for the moment, so the moment actually goes in the opposite direction. The moment due to the 5 lb force tends to cause the segment of the beam on the left side of the section to exhibit a downward concavity, and that corresponds to a negative bending moment, according to the sign convention for bending moment.

6. Plot these equations on a plot on top of each other.

Note that because the shearing force is a constant, it must be of the same magnitude at any point along the beam. As a convention, the shearing force diagram is plotted above or below a line corresponding to the neutral axis of the beam, but a plus sign must be indicated if it is a positive shearing force, and a minus sign should be indicated if it is a negative shearing force. A way to check the answer is to ensure the reaction force brings the problem back to 0. The shear is -5 until the last moment when the reaction force of +5lb brings the force to 0.

Since the function for the bending moment is linear, the bending moment diagram is a straight line. Thus, it is enough to use the two principal values of bending moments determined at  $x = 0$  ft and at  $x = 3$  ft to plot the bending moment diagram. As a convention, negative bending moment diagrams are plotted below the neutral axis of the beam, while positive bending moment diagrams are plotted above the axis of the beam.



Notice the units are included in the axes.

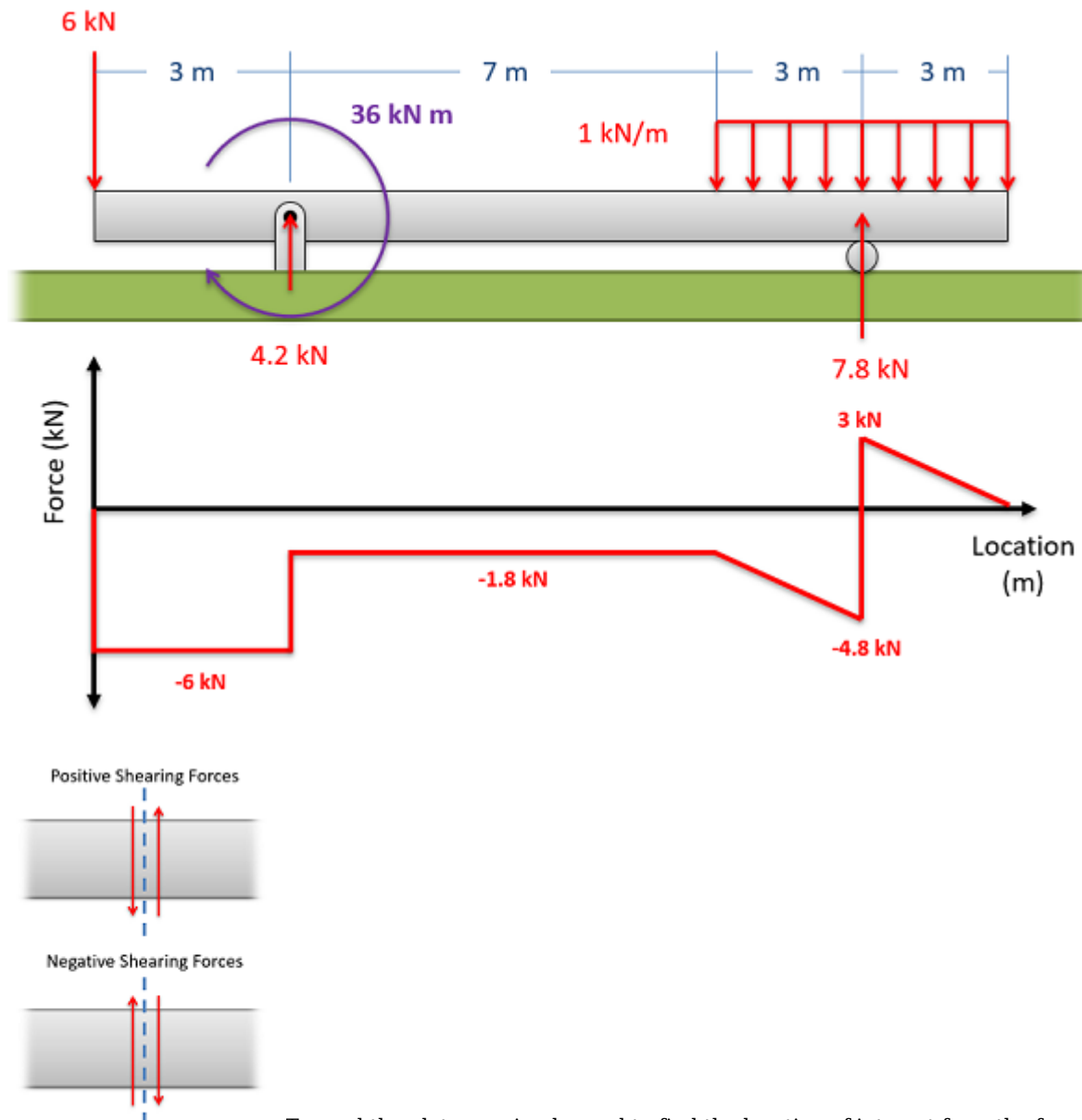
Here is a second explanation for how to create shear/moment diagrams:

#### Shear Diagram

To create the shear force diagram, we will use the following process.

1. Solve for all **external forces** acting on the body.
2. Draw out a **free body diagram** of the body **horizontally**. Leave all distributed forces as distributed forces and do not replace them with the equivalent point load.
3. Lined up below the free body diagram, draw a set of axes. The x-axis will represent the location (lined up with the free body diagram above), and the y-axis will represent the internal shear force.
4. Starting at zero at the right side of the plot, you will move to the right, pay attention to forces in the free body diagram above. As you move right in your plot, keep steady except...
  - Jump **upwards** by the magnitude of the force for any point forces going up.
  - Jump **downwards** by the magnitude of the force for any point forces going down.
  - For any **uniform distributed forces** you will have a **linear slope** where the magnitude of the distributed force is the slope of the line (positive slopes for upwards distributed forces, negative slopes for downwards distributed forces).
  - For **non-uniform distributed forces**, the shape of the shear diagram plot will be the **integral of the force function**.
  - You can ignore any moments or horizontal forces applied to the body.

By the time you get to the left end of the plot, you should always wind up coming **back to zero**. If you don't wind up back at zero, go back and check your previous work.



To read the plot, you simply need to find the location of interest from the free body diagram above, and read the corresponding value on the y-axis from your plot. Positive numbers represent an upwards internal shearing force to the right of the cross section and a downwards force on the left, and negative numbers indicate a downwards internal shearing force to the right of the cross section and a upwards force on the left. A visual of these forces can be seen in the diagram to the right.

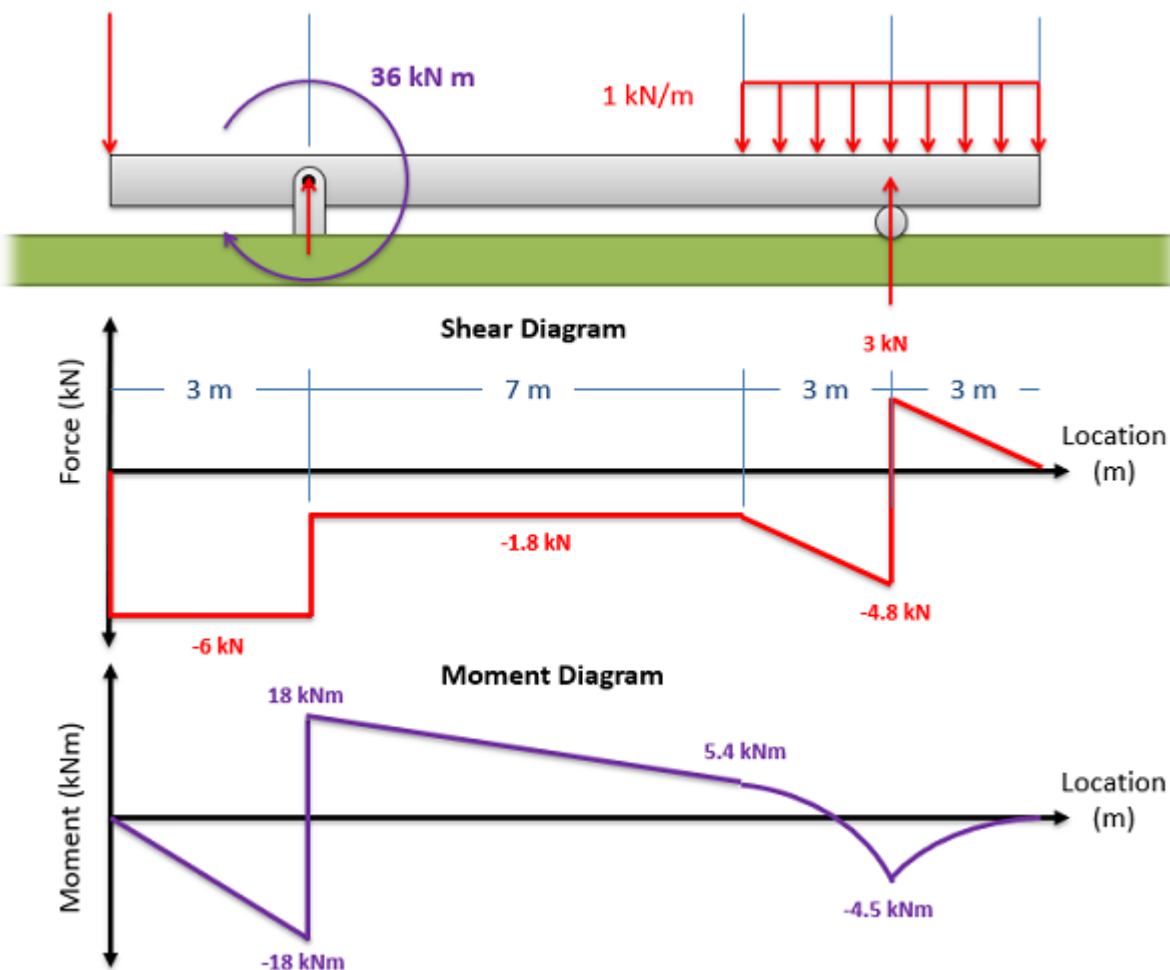
### Moment Diagram

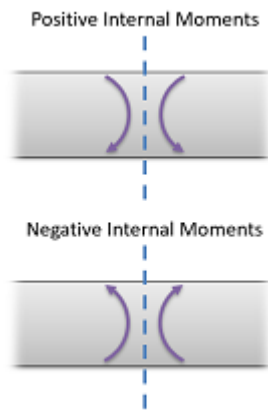
The moment diagram will plot out the internal bending moment within a horizontal beam that is subjected to multiple forces and moments perpendicular to the length of the beam. For practical purposes, this diagram is often used in the same circumstances as the shear diagram, and generally both diagrams will be created for analysis in these scenarios.

To create the moment diagram for a shaft, we will use the following process.

1. Solve for all external forces and moments, create a free body diagram, and create the shear diagram.
2. Lined up below the shear diagram, draw a set of axes. The x-axis will represent the location (lined up with the shear diagram and free body diagram above), and the y-axis will represent the internal bending moment.
3. Starting at zero at the right side of the plot, you will move to the right, pay attention to shear diagram and the moments in the free body diagram above. As you move right in your plot, the moment diagram will primarily be the **integral of the shear diagram**, except...
  - Jump **upwards** by the magnitude of the moment for any negative (**clockwise**) moments.
  - Jump **downwards** by the magnitude of the moment for any positive (**counter-clockwise**) moments.
  - You can ignore any forces in the free body diagram.

By the time you get to the left end of the plot, you should always wind up coming **back to zero**. If you don't wind up back at zero, go back and check your previous work.





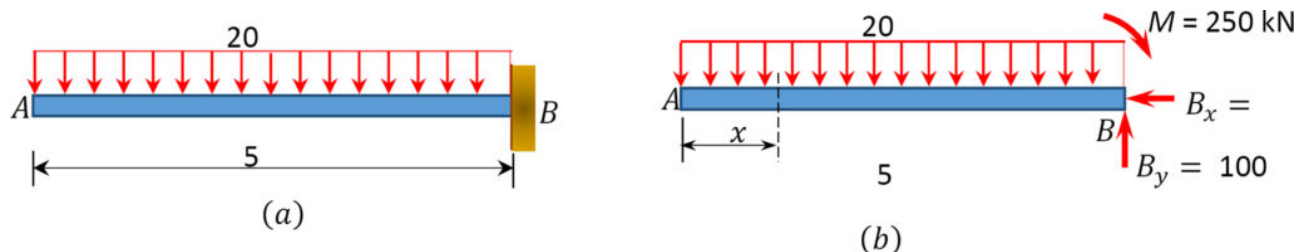
To read the plot, you simply need to take the find the location of interest from the free body diagram above, and read the corresponding value on the y-axis from your plot. Positive internal moments would cause the beam to bow downwards (think a smile shape) negative internal moments will cause the beam to bow upwards (think a frown shape). You can also see the positive and negative internal moments in the figure to the right.

Source: Engineering Mechanics, Jacob Moore, et al.

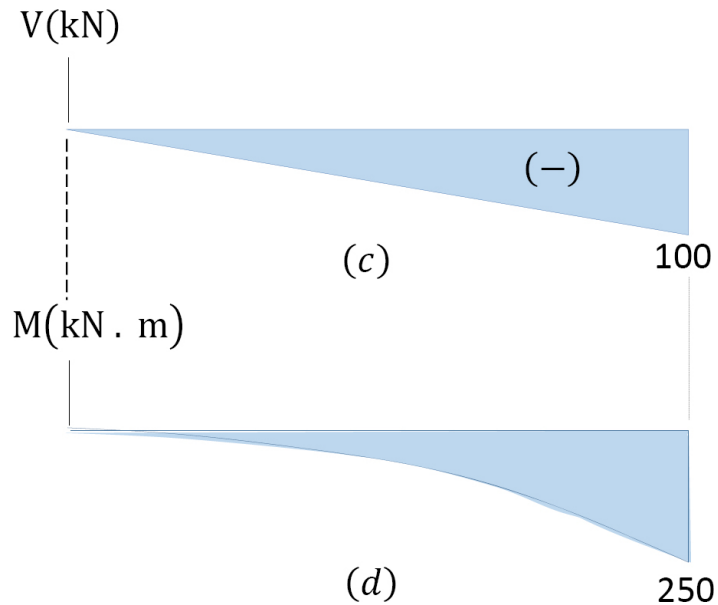
[http://mechanicsmap.psu.edu/websites/6\\_internal\\_forces/6-4\\_shear\\_moment\\_diagrams/shear\\_moment\\_diagrams.htm](http://mechanicsmap.psu.edu/websites/6_internal_forces/6-4_shear_moment_diagrams/shear_moment_diagrams.htm)  
1

## Example 2

Draw the shearing force and bending moment diagrams for the cantilever beam subjected to a uniformly distributed load in its entire length, as shown in [Figure 4.5a](#).



Answer:



*Support reactions.*

First, compute the reactions at the support. Since the support at  $B$  is fixed, there will possibly be three reactions at that support, namely  $B_y$ ,  $B_x$ , and  $M_B$ , as shown in the free-body diagram in [Figure 4.4b](#). Applying the conditions of equilibrium suggests the following:

$$\sum M_B = 0: \quad (20 \text{ kN/m})(5 \text{ m})(2.5 \text{ m}) - M = 0$$

$$M = 250 \text{ kN.m}$$

$$\sum F_y = 0: \quad -(20 \text{ kN/m})(5) + B_y = 0$$

$$B_y = 100 \text{ kN}$$

$$\sum F_x = 0: \quad B_x = 0$$

*Shear Force Function*

Let  $x$  be the distance of an arbitrary section from the free end of the cantilever beam, as shown in [Figure 4.5b](#). The shearing force of all the forces acting on the segment of the beam to the left of the section, as shown in [Figure 4.5e](#), is determined as follows:

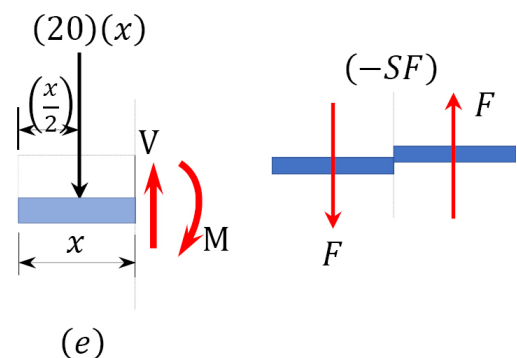
$$0 < x < 5$$

$$V = -20x$$

$$\text{When } x = 0, V = 0$$

$$\text{When } x = 2.5 \text{ m, } V = -50 \text{ kN}$$

$$\text{When } x = 5 \text{ m, } V = -100 \text{ kN}$$



The obtained expression is valid for the entire beam. The negative sign indicates a negative shearing force, which

was established from the sign convention for a shearing force. The expression also shows that the shearing force varies linearly with the length of the beam.

Shearing force diagram. Note that because the expression for the shearing force is linear, its diagram will consist of straight lines. The shearing force at  $x = 0$  m and  $x = 5$  m were determined and used for plotting the shearing force diagram, as shown in [Figure 4.5c](#). As shown in the diagram, the shearing force varies from zero at the free end of the beam to 100 kN at the fixed end. The computed vertical reaction of  $B_y$  at the support can be regarded as a check for the accuracy of the analysis and diagram.

### Bending Moment Function

The expression for the bending moment at a section of a distance  $x$  from the free end of the cantilever beam is as follows:

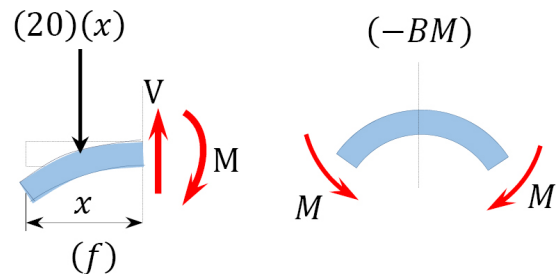
$$0 < x < 5 \text{ m}$$

$$M = -\frac{20x^2}{2}$$

$$\text{When } x = 0, M = 0$$

$$\text{When } x = 2.5 \text{ m}, M = -62.5 \text{ kN}\cdot\text{m}$$

$$\text{When } x = 5 \text{ m}, M = -250 \text{ kN}\cdot\text{m}$$



The negative sign indicates a negative moment, which was established from the sign convention for moment. As seen in [Figure 4.5f](#), the moment due to the distributed load tends to cause the segment of the beam on the left side of the section to exhibit an upward concavity, and that corresponds to a negative bending moment, according to the sign convention for bending moment.

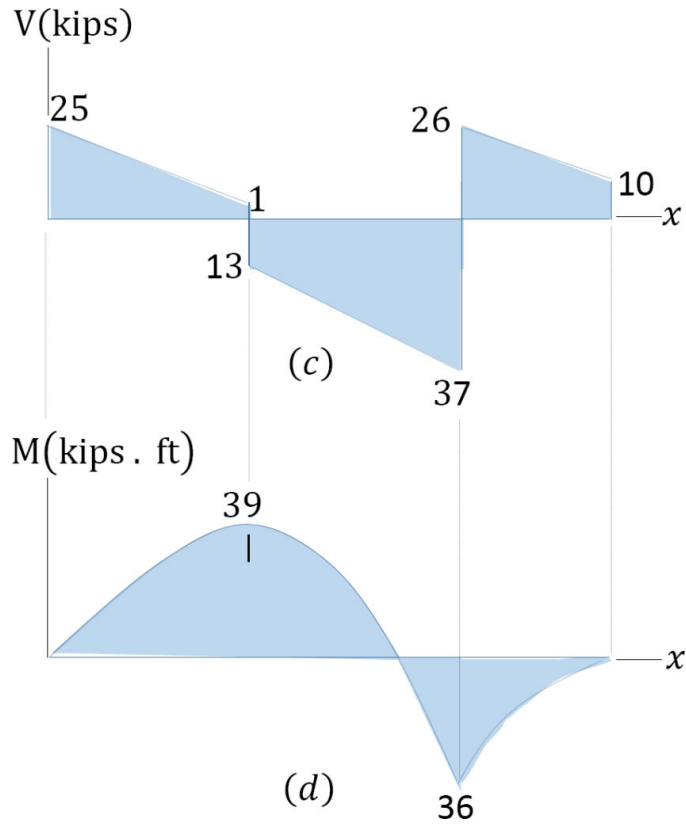
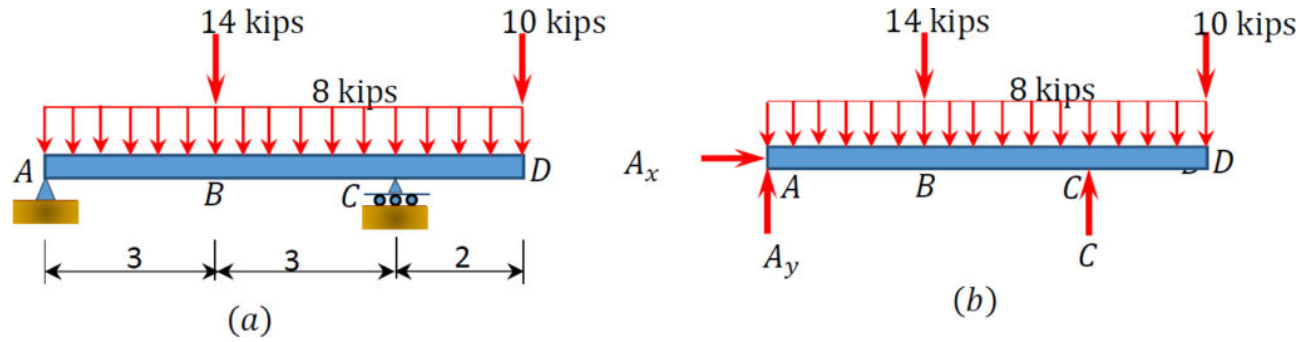
Bending moment diagram. Since the function for the bending moment is parabolic, the bending moment diagram is a curve. In addition to the two principal values of bending moment at  $x = 0$  m and at  $x = 5$  m, the moments at other intermediate points should be determined to correctly draw the bending moment diagram. The bending moment diagram of the beam is shown in [Figure 4.5d](#).

Source: Internal Forces in Beams and Frames,

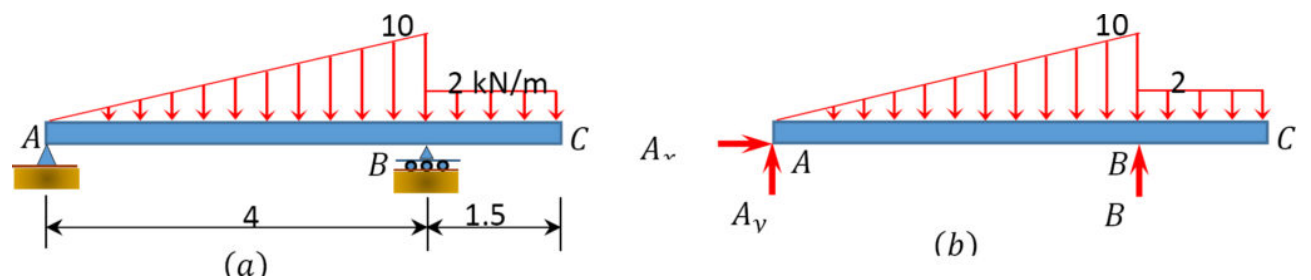
Libretexts. [https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames)

The following examples show the shear and moment diagrams for each beam. For details on how to solve each, go to: [https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames)

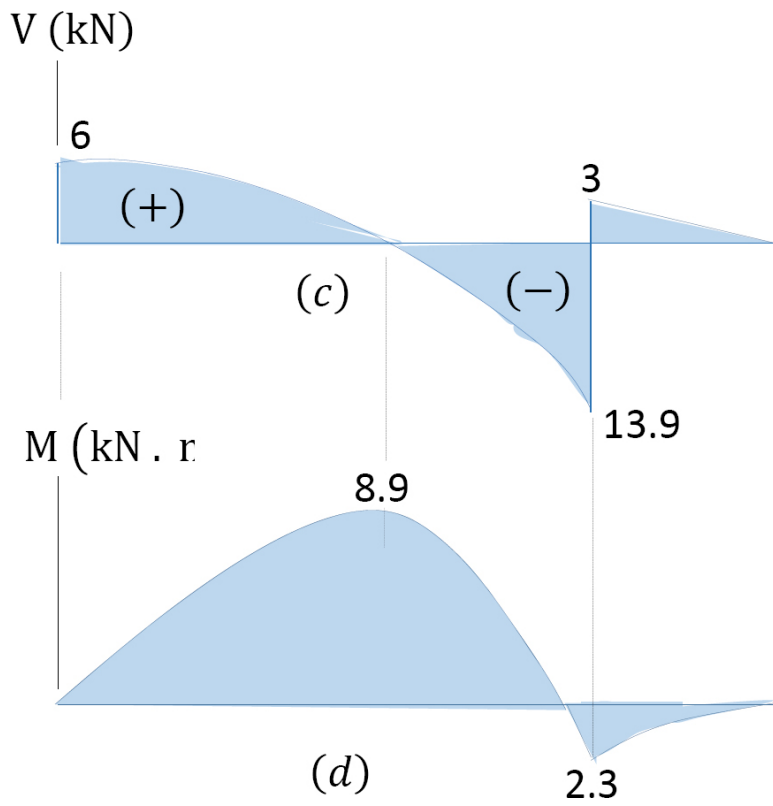
Example 3

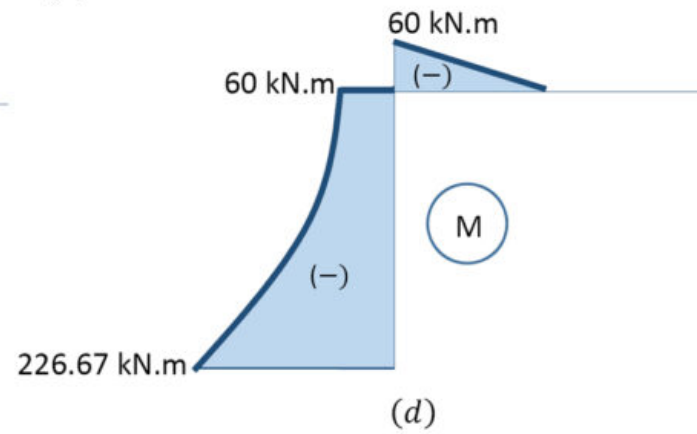
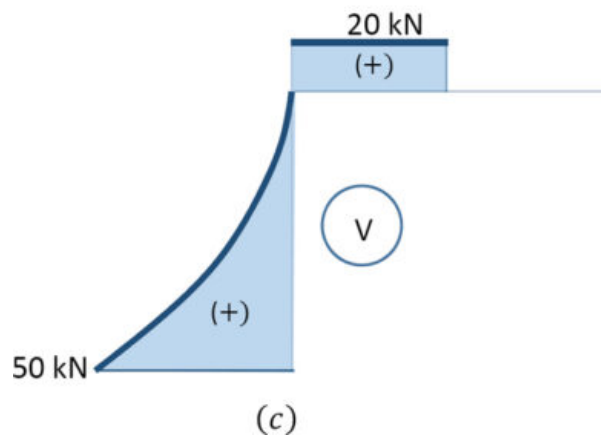
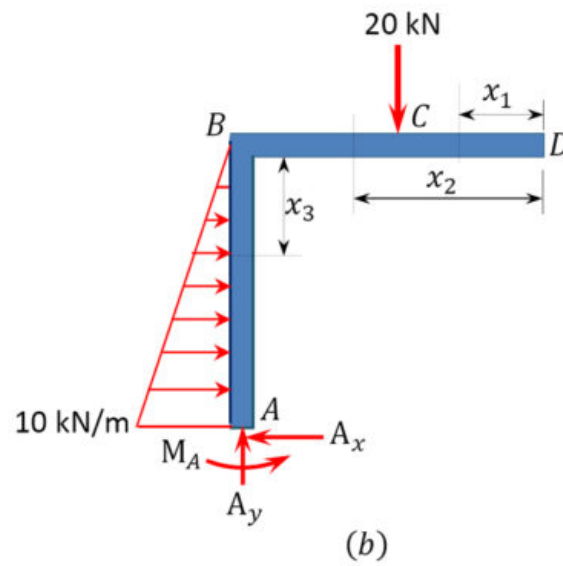
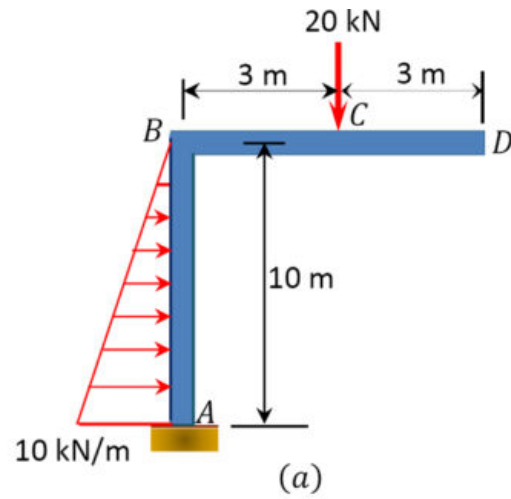


Example 4

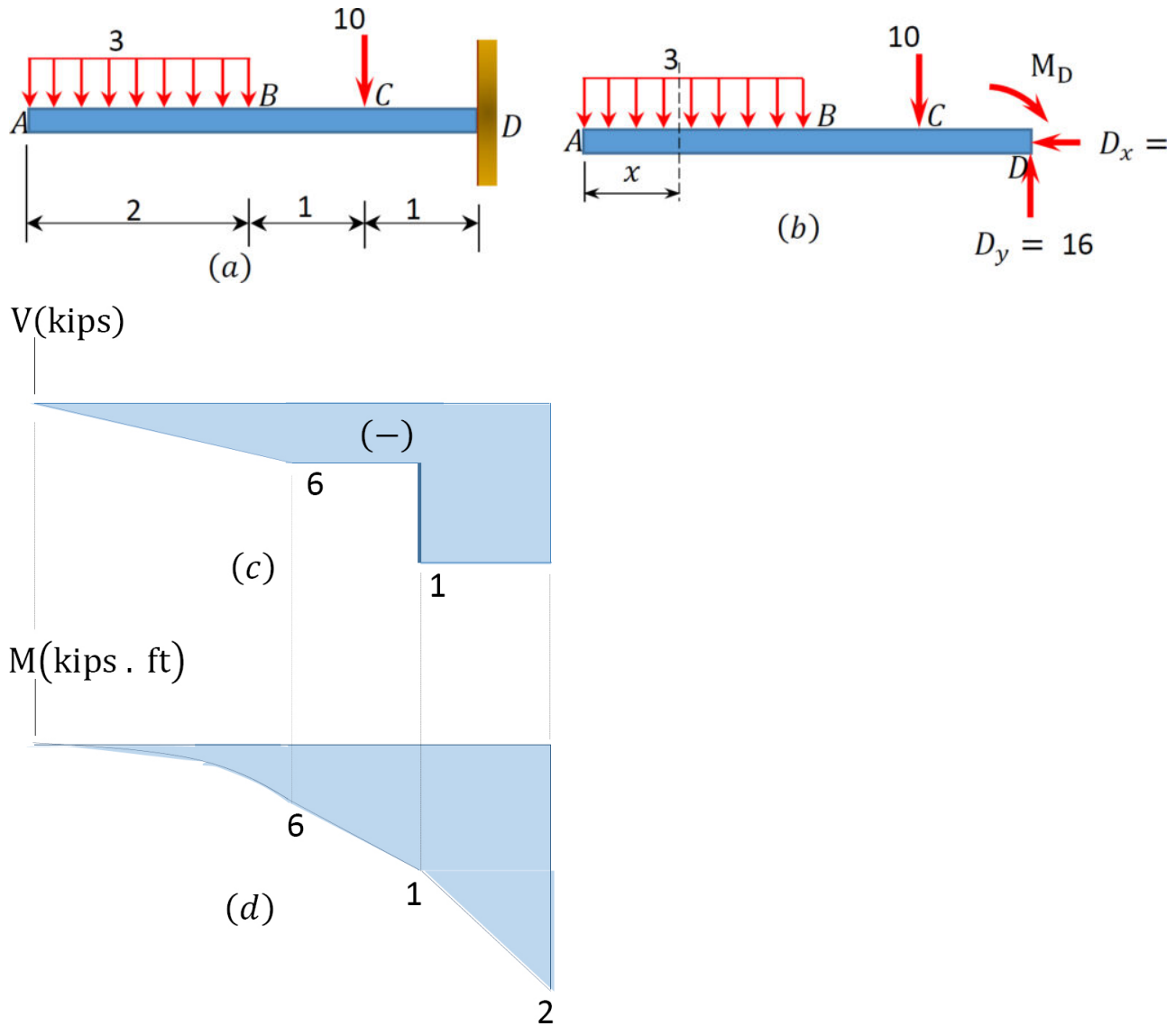




*Example 5*



Example 6



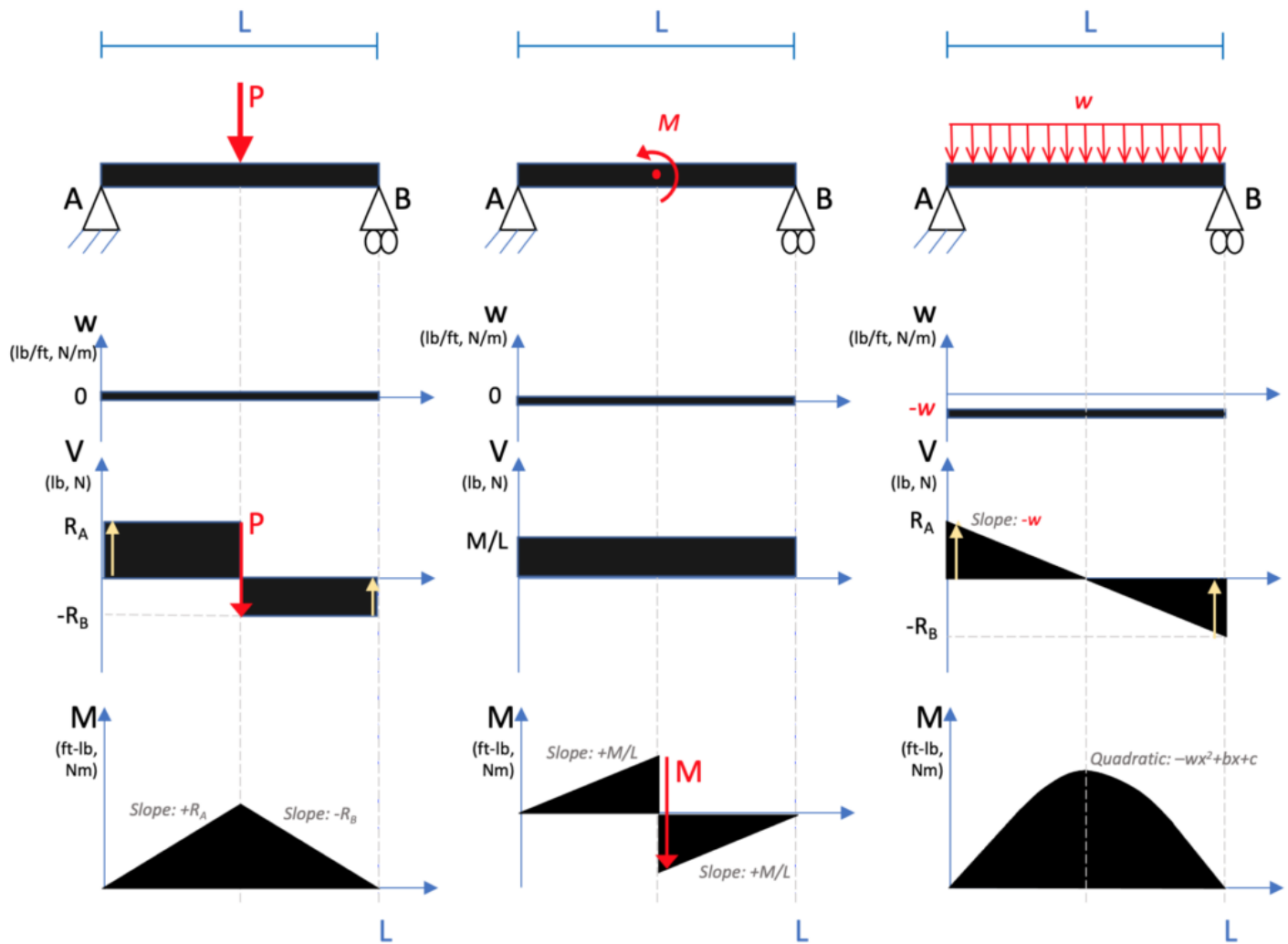
Source: Internal Forces in Beams and Frames,  
 Libretexts. [https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames)

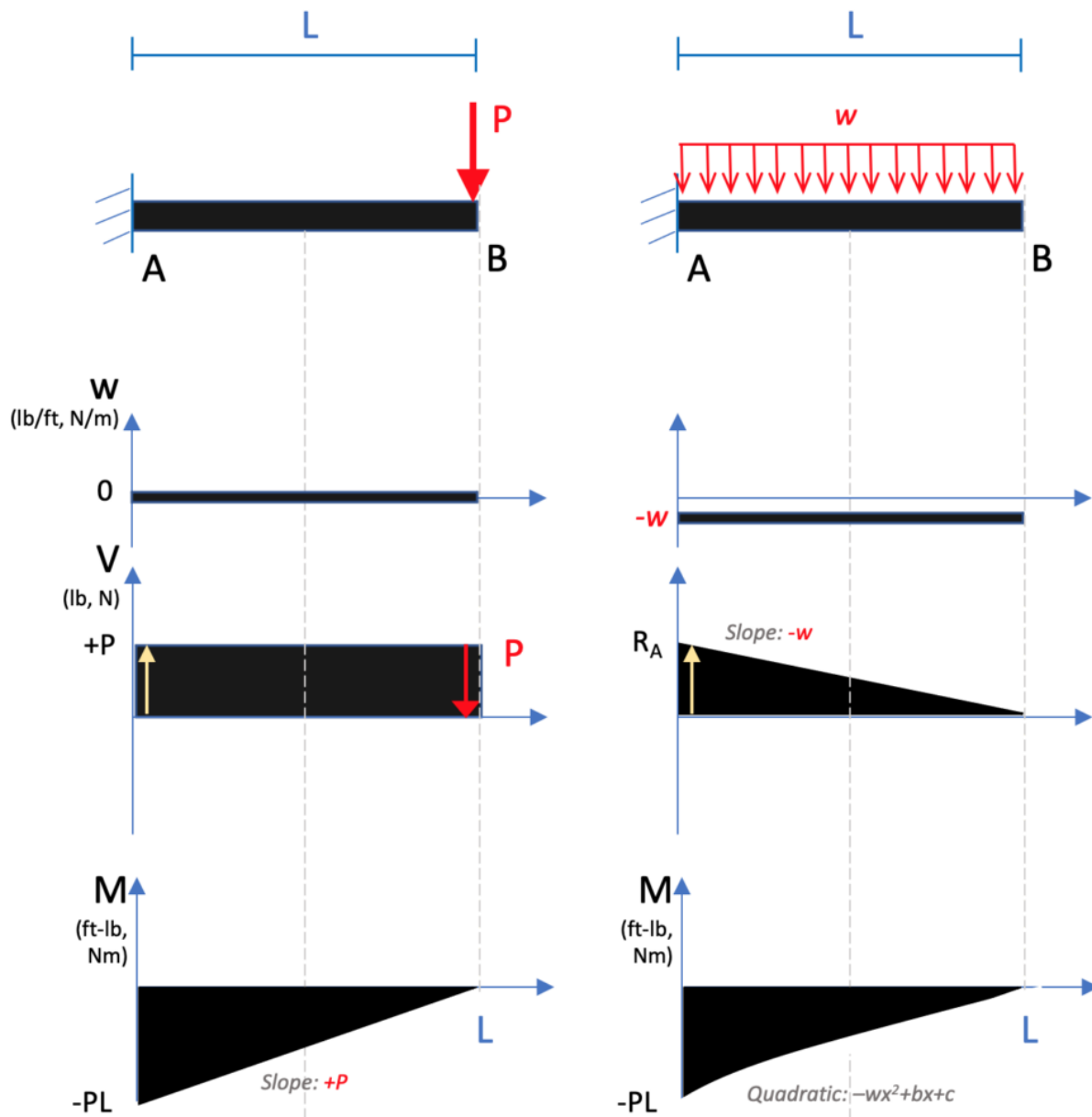
## 6.2.4 Tips & Plot Shapes

Though there are exceptions, these rules are generally true:

- +V means increasing M
- -V means decreasing M

- When  $V = 0$ , that's max or min  $M$





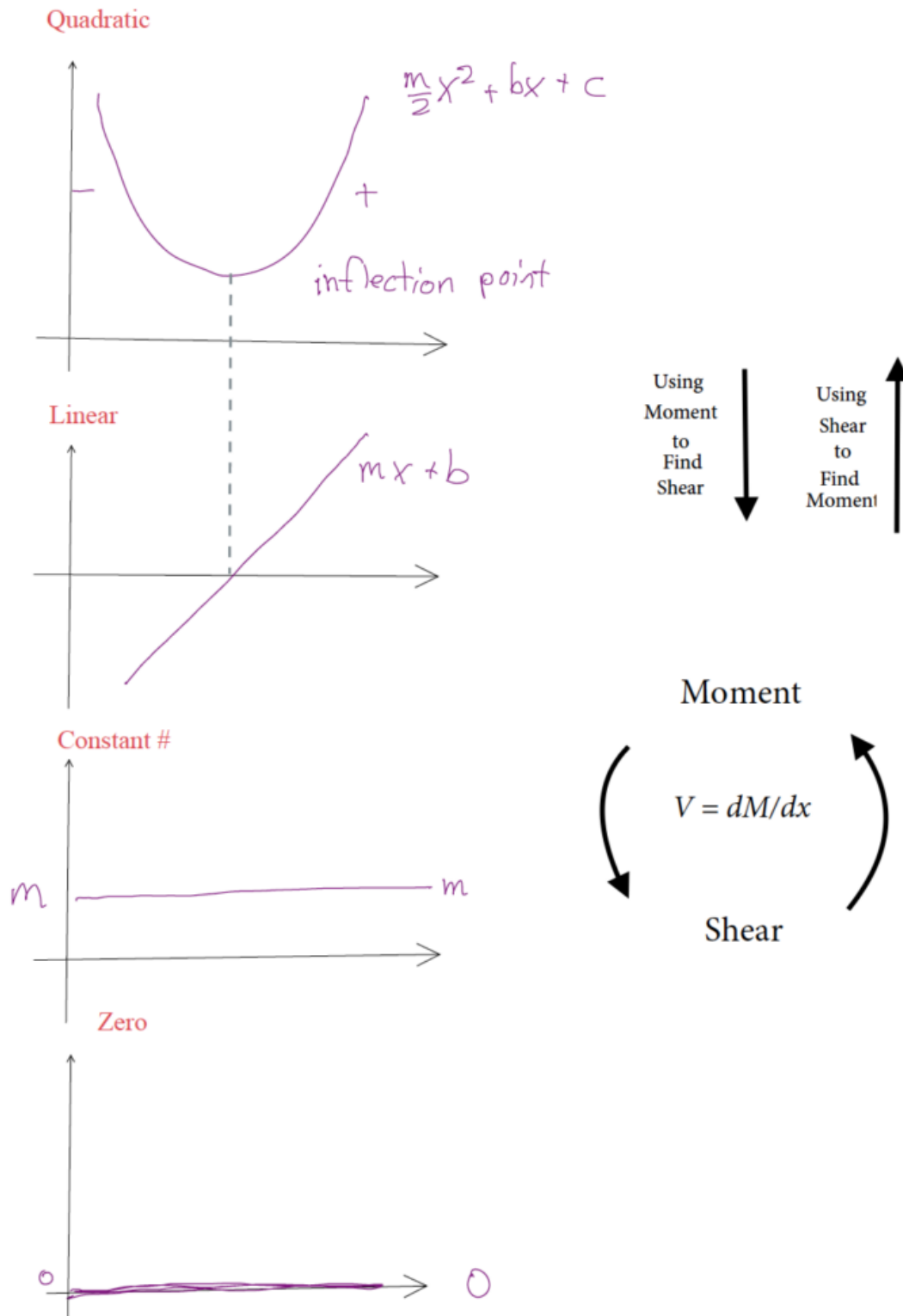
How does each plot start/end? Reactions only if no applied load/moment at ends:

- Cantilever:
  - At start/reaction: Nonzero  $V$  and  $M$
  - At end/unsupported end:  $0$  for both
- Simply supported
  - For  $V$ : Start and end with reaction forces
  - For  $M$ : Start and end at zero
- Where are the 'jumps' or inflection point where lines change?

- In V, forces 'jump' up or down where applied forces are, matching the direction they are applied (also reactions)
- In M, moments jump up or down where applied moments are, matching the direction
- Relationship between graphs
  - When there is an increasing slope in M, then Shear should be positive
  - When there is a decreasing slope in M, then Shear should be negative
  - When V is positive, M should be increasing
  - When V is negative, M should be decreasing
  - When intensity is positive, V should be increasing
  - When intensity is negative, V should be decreasing
  - Inflection points in the M plot (where the slope of the line changes from negative to positive & max/min values) should be 0 in the V plot
  - A zero value in the V plot should produce a max or min value in the M plot

The following figure shows the relationship between the derivatives. Remember that the derivative of  $x^2$  (quadratic) =  $x$  (linear). The derivative of  $x$  (linear) is a constant number. The derivative of a constant number is 0. The derivative of moment is shear, so if you have the shape of the moment, use this figure to approximate the shape of shear by going down the plots.

The reverse is true when going from shear to moment. The integral of shear is moment. The integral of 0 is a constant number. The integral of a constant number is linear. The integral of linear is quadratic. (The integral of quadratic is cubic). This progression moves up the plots from the bottom to the top.



There are a few online programs that can help confirm the shape that you found or help you learn how to translate loads into shear and moment diagrams. These are not acceptable to use on the exam or in homework and have limited free versions. This is not an endorsement of any of the sites, just showing

learning tools.

- <https://skyciv.com/free-beam-calculator/>
- <https://clearcalcs.com/freetools/beam-analysis/au>
- <https://beamguru.com/beam/>

#### Key Takeaways

*Basically:* Shear / Moment diagrams graphically display the internal loads along a beam.

*Application:* This can help you identify the major stress points to provide a safer design.

*Looking Ahead:* You will use this more in your structures class.



## 6.3 Examples

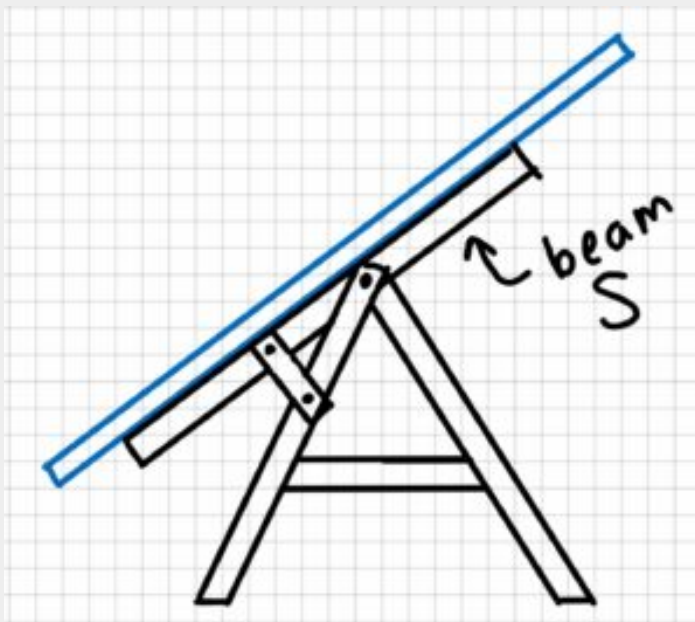
Here are examples from Chapter 6 to help you understand these concepts better. These were taken from the real world and supplied by FSDE students in Summer 2021. If you'd like to submit your own examples, please send them to the author [eosgood@upei.ca](mailto:eosgood@upei.ca).

### Example 6.3.1: Internal Forces - Submitted by Emma Christensen

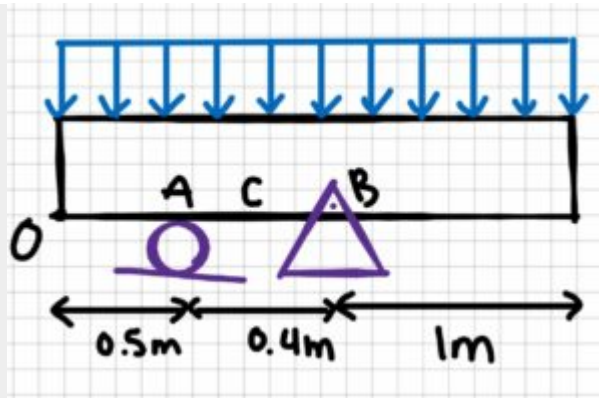
#### 1. Problem

The setup that holds the solar panels at the UPEI FSDE is modeled below. Considering beam S (1.9 m length), find the internal forces at point C. Assume the intensity of the solar panel on the beam is 200 N/m.

Sketch:

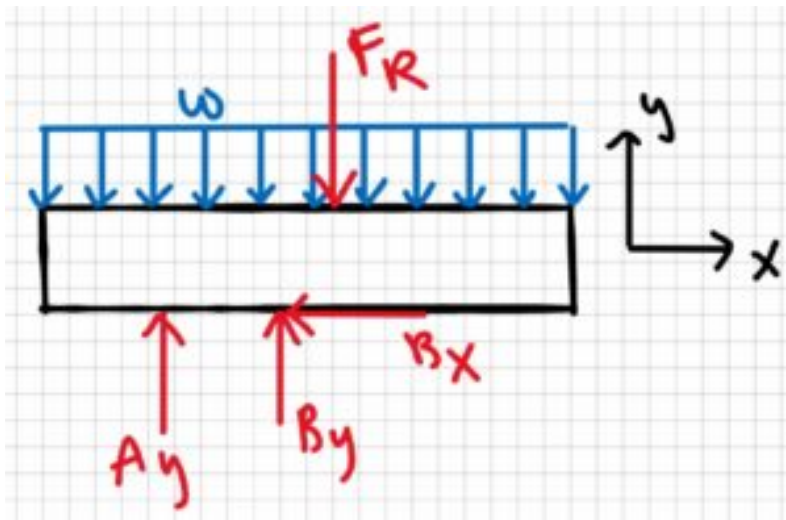


Model:



## 2. Draw

Free-body diagram:



## 3. Knowns and Unknowns

Knowns

- $w = 220 \text{ N/m}$
- $OA = 0.5 \text{ m}$
- $AC = 0.2 \text{ m}$
- $AB = 0.4 \text{ m}$
- $L = 1.9 \text{ m}$

Unknowns:  $N_c$ ,  $V_c$ ,  $M_c$

## 4. Approach

Use equilibrium equations. First solve for reaction forces, then make a cut at C and solve for the internal forces.

## 5. Analysis

$$w = \frac{F}{L} \quad F = wL \quad F_R = 220 \text{ N/m} \cdot 1.9 \text{ m} \quad F_R = 418 \text{ N} \quad \sum F_X = 0 = B_X$$

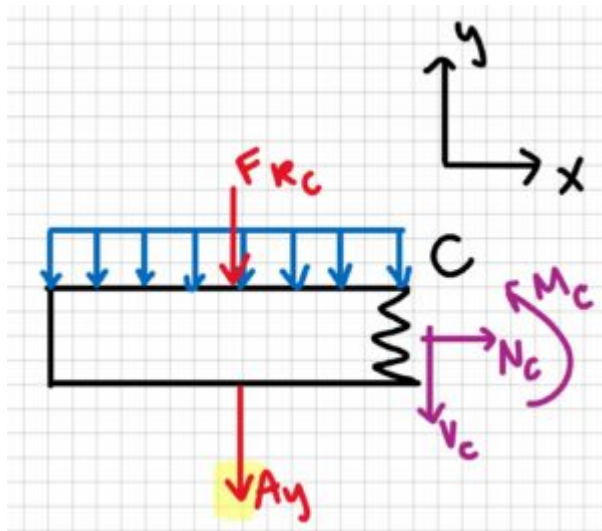
Find reaction forces:

$$\sum M_A = 0 = B_y(0.4 \text{ m}) - F_R(0.55 \text{ m}) \quad (0.4 \text{ m}) B_y = 418 \text{ N}(0.55 \text{ m}) \quad B_y = \frac{229.9 \text{ N} \cdot \text{m}}{0.4 \text{ m}} \quad B_y = 574.75 \text{ N}$$

$$\sum F_Y = 0 = -F_R + A_y + B_y \quad A_y = F_R - B_y \quad A_y = 418 \text{ N} - 574.74 \text{ N} \quad A_y = -156.75 \text{ N}$$

The answer we got for  $A_y$  is negative, which means that the arrow should be drawn in the other direction. We will change it for our next sketch.

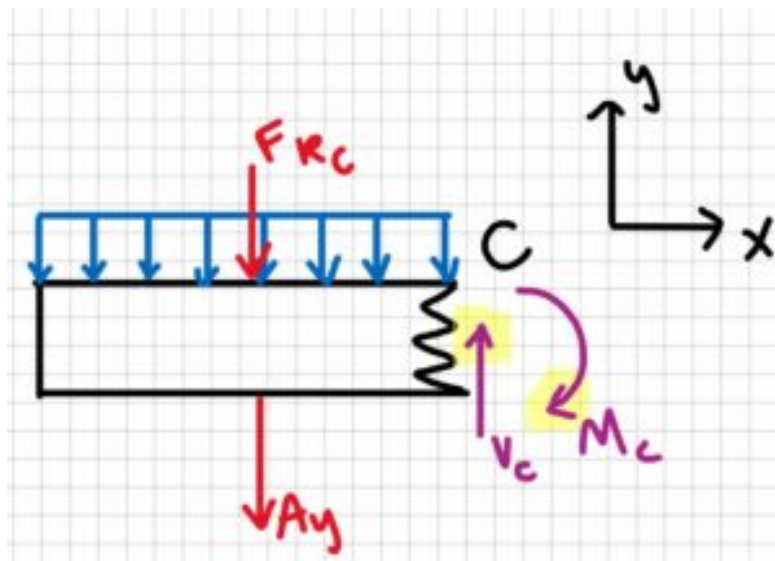
Make a cut at C:



Now solve for the internal forces:

$$\sum F_X = 0 \Rightarrow N_C = 0 \quad \sum F_Y = 0 = -A_y - V_C - (w \cdot L) \quad V_C = -156.75 \text{ N} - (220 \text{ N/m} \cdot 0.95 \text{ m}) \quad V_C = -346.75 \text{ N} \quad \sum M_C = A_y(0.2 \text{ m}) + M_C + (F_{Rc} \cdot 0.475 \text{ m}) \quad M_C = -156.75 \text{ N} \cdot (0.2 \text{ m}) - (220 \text{ N/m} \cdot 0.95 \text{ m} \cdot 0.475 \text{ m}) \quad M_C = -130.625 \text{ N} \cdot \text{m}$$

Final FBD, showing the arrows in the correct directions:



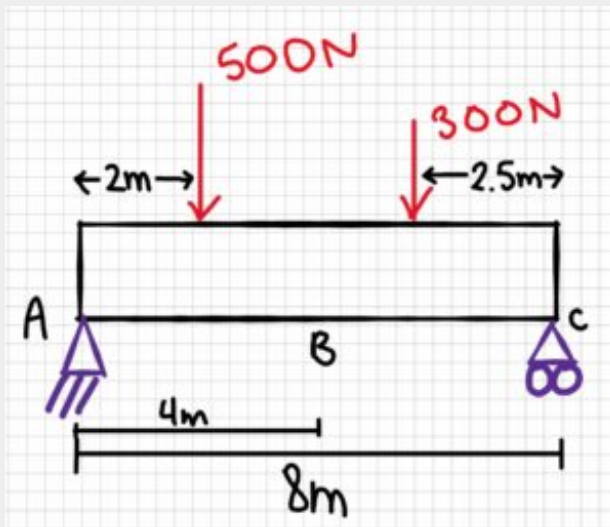
## 6. Review

It makes sense that  $A_y$  and  $B_y$  are in different directions, because the resultant force  $F_r$  of the solar panel on the beam is not between A and B. It also makes sense that the moment at C is in the clockwise direction rather than the counterclockwise directions, when you think about the direction of the forces applied to the beam.

# Example 6.3.2: Shear/Moment Diagrams - Submitted by Deanna Malone

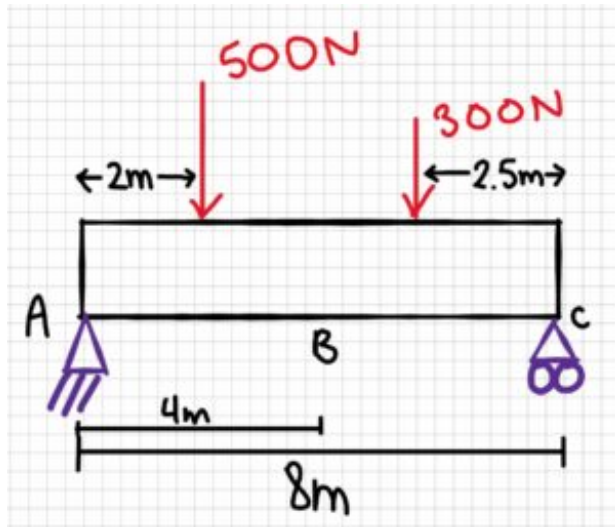
## 1. Problem

A beam that is simply supported has two point loads acting on it. One acts 2 m from point A and the other acts at 2.5 m from C. Point B is in the middle of the beam. The first point load is 500 N and the second is 300 N. What are the internal forces at point B? Solve for reaction forces and include a shear/moment diagram.

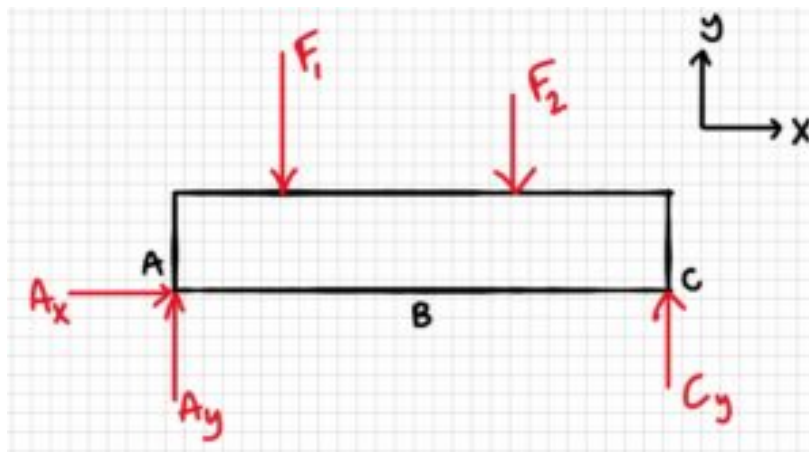


## 2. Draw

Sketch:



Free-body diagram:



### 3. Knowns and Unknowns

Knowns:

- $F_1 = 500 \text{ N}$
- $F_2 = 300 \text{ N}$

Unknowns:  $A_y$ ,  $A_x$ ,  $C_y$ ,  $V_B$ ,  $M_B$ ,  $N_B$

### 4. Approach

Shear/moment equations, EOM equations

### 5. Analysis

Solve for reaction forces:

$(A_x, C_y)$

$\begin{aligned} \sum F_x = 0 \end{aligned}$

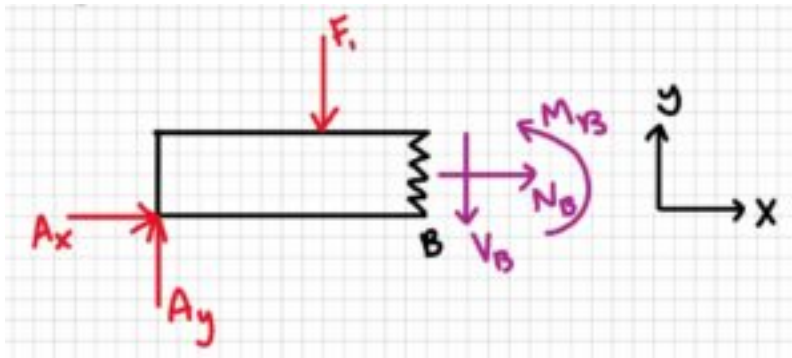
$\sum F_x = 0 \Rightarrow A_x = 0$

$$\begin{aligned} \sum M_A &= 0 = -F_1 \cdot 2 \text{ m} - F_2 \cdot 5.5 \text{ m} + C_y \cdot 8 \text{ m} \\ C_y &= F_1 \cdot 2 \text{ m} + F_2 \cdot 5.5 \text{ m} \\ C_y &= \frac{500 \text{ N} \cdot 2 \text{ m} + 300 \text{ N} \cdot 5.5 \text{ m}}{8 \text{ m}} \\ C_y &= 331.25 \text{ N} \end{aligned}$$

(A<sub>y</sub>)

$$\begin{aligned} \sum F_y &= 0 = A_y + C_y - F_1 - F_2 \\ A_y &= F_1 + F_2 - C_y \\ A_y &= 500 \text{ N} + 300 \text{ N} - 331.25 \text{ N} \\ A_y &= 468.75 \text{ N} \end{aligned}$$

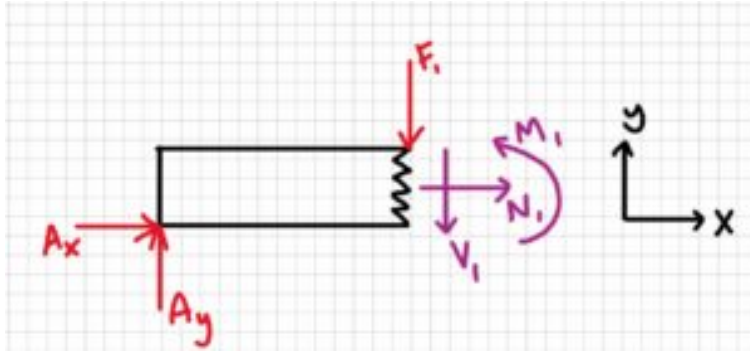
Cut 1: at B



$$\begin{aligned} \sum F_x &= 0 = A_x + N_B = 0 \\ N_B &= 0 \\ \sum F_y &= 0 = A_y - V_B - F_1 \\ V_B &= A_y - F_1 \\ V_B &= 468.75 \text{ N} - 500 \text{ N} \\ V_B &= -31.25 \text{ N} \end{aligned}$$

$$\begin{aligned} \sum M_B &= 0 = -A_y(4 \text{ m}) + F_1(2 \text{ m}) + M_B \\ M_B &= A_y(4 \text{ m}) - F_1(2 \text{ m}) \\ M_B &= 468.75 \text{ N}(4 \text{ m}) - 500 \text{ N}(2 \text{ m}) \\ M_B &= 875 \text{ N} \cdot \text{m} \end{aligned}$$

Cut 2: At the point where F<sub>1</sub> is applied

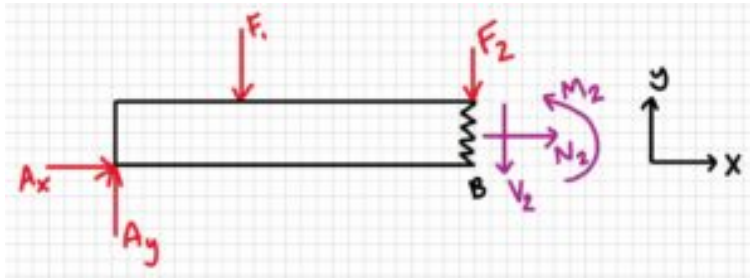


```

\begin{aligned}
&\sum M_1 = 0 \quad \&= -A_y(2\text{ m}) + M_1 = 0 \\
M_1 \quad \&= A_y(2\text{ m}) \\
M_1 \quad \&= 468.75\text{ N}(2\text{ m}) \\
M_1 \quad \&= 937.5\text{ N}\cdot\text{m} \\
\end{aligned}

```

Cut 3: At the point where  $F_2$  is applied

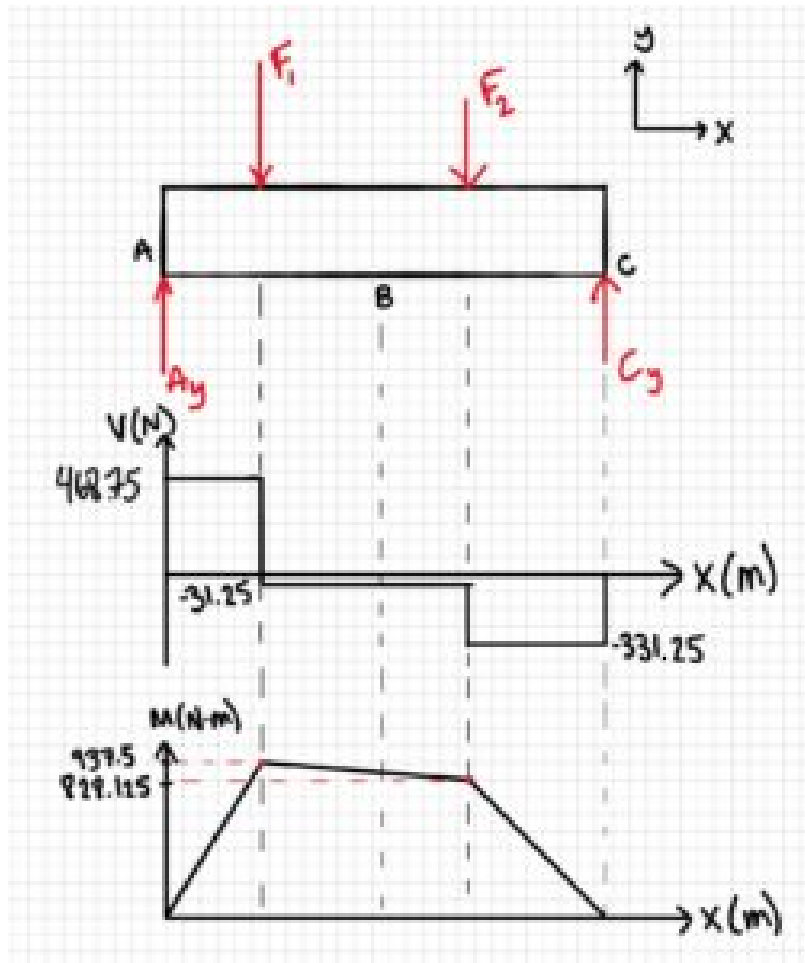


```

\begin{aligned}
&\sum M_2 = 0 \quad \&= -A_y(5.5\text{ m}) + F_1(3.5\text{ m}) + M_2 \\
M_2 \quad \&= A_y(5.5\text{ m}) - F_1(3.5\text{ m}) \\
M_2 \quad \&= 468.75\text{ N}(5.5\text{ m}) - 500\text{ N}(3.5\text{ m}) \\
M_2 \quad \&= 828.125\text{ N}\cdot\text{m} \\
\end{aligned}

```

Answer:  $N_B = 0$ ,  $V_B = -31.25\text{ N}$ ,  $M_B = 875\text{ Nm}$



### 6. Review

The reaction forces make sense as they offset the applied forces. The shear/moment diagrams returned to zero so they are correct too. The moment found at B is in the moment diagram, it is smaller than the maximum.



# CHAPTER 7: INERTIA

This chapter is a major preparation for objects that rotate during dynamics class. We will cover cg, inertia, composite shapes, and rotating about axes other than the centre of mass (cm) using the parallel axis theorem. Here are the sections in this Chapter:

- 7.1 Center of Mass: Single Objects - How to find the cm of a single object
- 7.2 Center of Mass: Composite Shapes - Finding the cm for multiple objects or complex objects
- 7.3 Types of Inertia - Wait - there's more than one type of inertia!
- 7.4 Mass Moment of Inertia - How to calculate the mass moment of inertia for rotational motion for single objects
- 7.5 Inertia Intro: Parallel Axis Theorem - Calculating the inertia for single objects rotating about a different axis or for composite problem
- 7.6 Examples - Examples from your peers

Here are the important equations for this chapter:

## Ch 7 Equations

### Center of Mass

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i} \quad \bar{y} = \frac{\sum m_i y_i}{\sum m_i} \quad \bar{z} = \frac{\sum m_i z_i}{\sum m_i}$$

Area =  $bh$   
 $r_{cm} = \left[ \frac{b}{2}, \frac{h}{2} \right]$

Area =  $\pi(r_o^2 - r_i^2)$   
 $r_o$  = outer radius  
 $r_i$  = inner radius

Area =  $\frac{\pi}{2}r^2$   
 $r_{cm} = \left[ 0, \frac{4}{3\pi}r \right]$

Area =  $\frac{\pi}{4}r^2$   
 $r_{cm} = \left[ \frac{4}{3\pi}r, \frac{4}{3\pi}r \right]$

Area =  $\frac{1}{2}bh$   
 $r_{cm} = \left[ \frac{b}{3}, \frac{h}{3} \right]$

Area =  $\pi ab$

### Additional Concepts

- Area Moment of Inertia
- Product Moment of Inertia
- Radius of gyration  $k = I/m$

Density =  $\frac{\text{mass}}{\text{Volume}}$

### Mass Moment of Inertia (MMOI)

$$I = \int_m r^2 dm \quad I = \sum m_i r_i^2$$

### Parallel Axis Theorem

$$I_O = I_G + (m)(d)^2$$

### Composites Bodies

1. Determine type of shape
2. Determine which axis to rotate about
3. Find center of mass (G) for each shape
4. Find  $I_G$  for each shape
5. Find distance from cm to axis of rotation
6. Use parallel axis theorem to find  $I_O$  for each shape
7. Add up all of the  $I_O$ 's

$$I_G = I_G + (m)(d)^2$$

$I_{xx} = \frac{1}{12}mh^2$   
 $I_{yy} = \frac{1}{12}m(h^2 + b^2)$   
 $I_{zz} = \frac{1}{12}mb^2$

$I_{xx} = \frac{1}{12}m(h^2 + d^2)$   
 $I_{yy} = \frac{1}{12}m(d^2 + w^2)$   
 $I_{zz} = \frac{1}{12}m(h^2 + w^2)$

$I_{xx} = I_{zz} = \frac{1}{2}mr^2$   
 $I_{yy} = mr^2$

$I_{xx} = I_{zz} = \frac{1}{4}mr^2$   
 $I_{yy} = \frac{1}{2}mr^2$   
 $I_{yy'} = \frac{3}{2}mr^2$

$I_{xx} = I_{zz} = \frac{1}{12}m(3r^2 + h^2)$   
 $I_{yy} = \frac{1}{2}mr^2$

$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mr^2$

$I_{xx} = I_{zz} = \frac{1}{12}ml^2$   
 $I_{yy} = 0$   
 $I_{xx'} = I_{zz'} = \frac{1}{3}ml^2$

# 7.1 Center of Mass: Single Objects

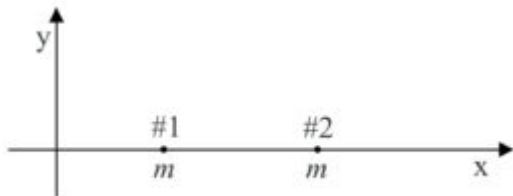
To start, let's calculate the center of mass! This is a weighted function, similar to when we found the location of the resultant force from multiple distributed loads and forces.

$$\bar{x} = \frac{m_1 x_1}{m_1 + m_2} + \frac{m_2 x_2}{m_1 + m_2}$$

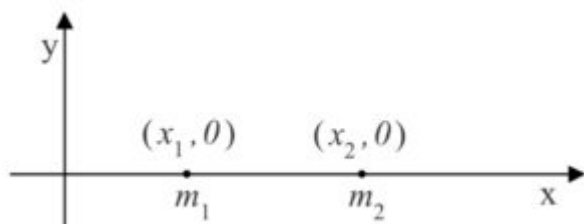
When the density is the same throughout a shape, the center of mass is also the centroid (geometric center).

## 7.1.1 Center of Mass of Two Particles

Consider two particles, having one and the same mass  $m$ , each of which is at a different position on the  $x$  axis of a Cartesian coordinate system.



Common sense tells you that the average position of the material making up the two particles is midway between the two particles. Common sense is right. We give the name “center of mass” to the average position of the material making up a distribution, and the center of mass of a pair of same-mass particles is indeed midway between the two particles. How about if one of the particles is more massive than the other? One would expect the center of mass to be closer to the more massive particle, and again, one would be right. To determine the position of the center of mass of the distribution of matter in such a case, we compute a weighted sum of the positions of the particles in the distribution, where the weighting factor for a given particle is that fraction, of the total mass, that the particle's own mass is. Thus, for two particles on the  $x$  axis, one of mass  $m_1$ , at  $x_1$ , and the other of mass  $m_2$ , at  $x_2$ ,



the position  $x$  of the center of mass is given by equation 8-1:

$$\bar{x} = \frac{m_1 x_1}{m_1 + m_2} + \frac{m_2 x_2}{m_1 + m_2}$$

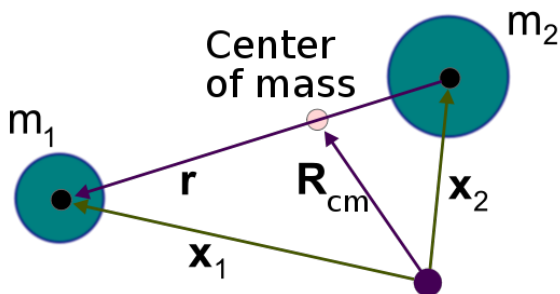
Note that each weighting factor is a proper fraction and that the sum of the weighting factors is always 1. Also note that if, for instance,  $m_1$  is greater than  $m_2$ , then the position  $x_1$  of particle 1 will count more in the sum, thus ensuring that the center of mass is found to be closer to the more massive particle (as we know it must be). Further note that if  $m_1 = m_2$ , each weighting factor is  $1/2$ , as is evident when we substitute  $m$  for both  $m_1$  and  $m_2$  in equation 8-1:

$$\bar{x} = \frac{m}{m+m} x_1 + \frac{m}{m+m} x_2 \quad \bar{x} = \frac{1}{2} x_1 + \frac{1}{2} x_2 \quad \bar{x} = \frac{x_1 + x_2}{2}$$

The center of mass is found to be midway between the two particles, right where common sense tells us it has to be.

Source: Calculus-Based Physics 1, Jeffery W. Schnick. p142,  
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=ce74a181-ccde-491c-848d-05489ed182e7>

Below is a more visual representation of where the COM would be for two different weighing particles.

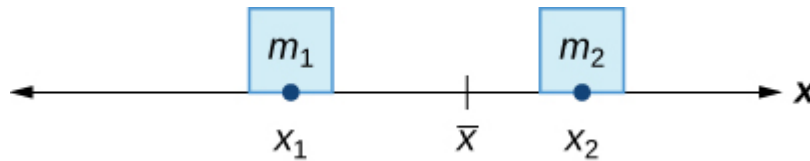


Source (image): Two\_body\_jacobi.svg: CWitte,  
 from JPG by Brews oharederivative work:  
 WillowW via Wikimedia Commons  
[https://zh.wikipedia.org/wiki/File:Jacobi\\_coordinates.svg](https://zh.wikipedia.org/wiki/File:Jacobi_coordinates.svg)

A second explanation:

The most common real-life example of a system like this is a playground seesaw, or teeter-totter, with children of different weights sitting at different distances from the center. On a seesaw, if one child sits at each end, the heavier child sinks down and the lighter child is lifted into the air. If the heavier child slides in toward the center,

though, the seesaw balances. Applying this concept to the masses on the rod, we note that the masses balance each other if and only if  $m_1 d_1 = m_2 d_2$ .



This idea is not limited just to two point masses. In general, if  $n$  masses,  $m_1, m_2, \dots, m_n$ , are placed on a number line at points  $x_1, x_2, \dots, x_n$ , respectively, then the center of mass of the system is given by:

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}$$

### Example 1:

Suppose four point masses are placed on a number line as follows:

- $m_1 = 30 \text{ kg}$ , placed at  $x_1 = -2 \text{ m}$
- $m_2 = 5 \text{ kg}$ , placed at  $x_2 = 3 \text{ m}$
- $m_3 = 10 \text{ kg}$ , placed at  $x_3 = 6 \text{ m}$
- $m_4 = 15 \text{ kg}$ , placed at  $x_4 = -3 \text{ m}$ .

#### Solution

Find the moment of the system with respect to the origin and find the center of mass of the system.

First, we need to calculate the moment of the system (the top part of the fraction):

$$M = \sum_{i=1}^4 m_i x_i = (30 \text{ kg})(-2 \text{ m}) + (5 \text{ kg})(3 \text{ m}) + (10 \text{ kg})(6 \text{ m}) + (15 \text{ kg})(-3 \text{ m}) \\ = (-60 + 15 + 60 - 45) \text{ kg} \cdot \text{m} = -30 \text{ kg} \cdot \text{m}$$

Now, to find the center of mass, we need the total mass of the system:

$$m = \sum_{i=1}^4 m_i = (30 + 5 + 10 + 15) \text{ kg} = 60 \text{ kg}$$

$$\bar{x} = \frac{M}{m} = \frac{-30 \text{ kg} \cdot \text{m}}{60 \text{ kg}} = -0.5 \text{ m}$$

The center of mass is located  $1/2 \text{ m}$  to the left of the origin.

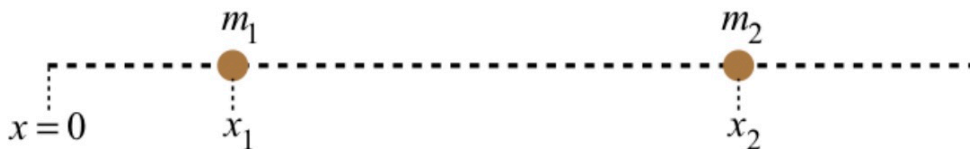
Source: "Moments and Centers of Mass" by LibreTexts, <https://eng.libretexts.org/@go/page/67237>

## 7.1.2 Center of Mass in 2D & 3D

When we are looking at multiple objects in 2D or 3D, we perform the center of mass equation multiple times in the x, y, and z directions.

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i} \quad \bar{y} = \frac{\sum_{i=1}^n m_i y_i}{\sum_{i=1}^n m_i} \quad \bar{z} = \frac{\sum_{i=1}^n m_i z_i}{\sum_{i=1}^n m_i}$$

In some sense, one can think about the center of mass of a single object as its “average position.” Let’s consider the simplest case of an “object” consisting of two tiny particles separated along the x-axis, as seen below:



If the two particles have equal mass, then it’s pretty clear that the “average position” of the two-particle system is halfway between them. If the masses of the two particles are different, would the “average position” still be halfway between them? Perhaps in some sense this is true, but we are not looking for a *geometric center*, we are looking for the average placement of mass. If \$m\_1\$ has twice the mass of \$m\_2\$, then when it comes to the average placement of mass, \$m\_1\$ gets “two votes.” With more of the mass concentrated at the position \$x\_1\$ than at \$x\_2\$, the center of mass should be closer to \$x\_1\$ than \$x\_2\$. We achieve the perfect balance by “weighting” the positions by the fraction of the total mass that is located there. Accordingly, we define as the center of mass:

$$\bar{x}_{cm} = \left( \frac{m_1}{m_1 + m_2} \right) x_1 + \left( \frac{m_2}{m_1 + m_2} \right) x_2 = \frac{m_1 x_1 + m_2 x_2}{M_{system}}$$

If there are more than two particles, we simply add all of them into the sum in the numerator. To extend this definition of center of mass into three dimensions, we simply need to do the same things in the y and z directions. A position vector for the center of mass of a system of many particles would then be:

$$\begin{aligned} \vec{r}_{cm} &= \bar{x}_{cm} \underline{\hat{i}} + \bar{y}_{cm} \underline{\hat{j}} + \bar{z}_{cm} \underline{\hat{k}} \\ &= \frac{[m_1 x_1 + m_2 x_2 + \dots] \underline{\hat{i}} + [m_1 y_1 + m_2 y_2 + \dots] \underline{\hat{j}} + [m_1 z_1 + m_2 z_2 + \dots] \underline{\hat{k}}}{M} \\ &= \frac{m_1 [x_1 \underline{\hat{i}} + y_1 \underline{\hat{j}} + z_1 \underline{\hat{k}}] + m_2 [x_2 \underline{\hat{i}} + y_2 \underline{\hat{j}} + z_2 \underline{\hat{k}}] + \dots}{M} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots}{M} \end{aligned}$$

Source: “Center of Mass” by Tom Weideman,  
[https://phys.libretexts.org/Courses/University\\_of\\_California\\_Davis/UCD%3A\\_Physics\\_9A\\_Classical\\_Mechanics/4%3A\\_Linear\\_Momentum/4.2%3A\\_Center\\_of\\_Mass](https://phys.libretexts.org/Courses/University_of_California_Davis/UCD%3A_Physics_9A_Classical_Mechanics/4%3A_Linear_Momentum/4.2%3A_Center_of_Mass)

## Example 2:

Suppose three point masses are placed in the x-y plane as follows (assume coordinates are given in meters):

- $m_1 = 2 \text{ kg}$  placed at  $(-1, 3)\text{m}$ ,
- $m_2 = 6 \text{ kg}$  placed at  $(1, 1)\text{m}$ , and
- $m_3 = 4 \text{ kg}$  placed at  $(2, -2)\text{m}$ .

Find the center of mass of the system.

### Solution

First we calculate the total mass of the system:

$$m = \sum_{i=1}^3 m_i = (2 + 6 + 4) \text{ kg} = 12 \text{ kg}$$

Next we find the moments with respect to the x- and y- axes:

$$M_x = \sum_{i=1}^3 m_i x_i \quad \text{quad} \quad = (2\text{kg})(-1\text{m}) + (6\text{kg})(1\text{m}) + (4\text{kg})(2\text{m}) \quad \text{quad} \quad = (-2+6+8)\text{kg}\cdot\text{m} \quad \text{quad} \quad = 12 \text{ kg}\cdot\text{m}$$

$$M_y = \sum_{i=1}^3 m_i y_i \quad \text{quad} \quad = (2\text{kg})(3\text{m}) + (6\text{kg})(1\text{m}) + (4\text{kg})(-2\text{m}) \quad \text{quad} \quad = (6+6-8)\text{kg}\cdot\text{m} \quad \text{quad} \quad = 4 \text{ kg}\cdot\text{m}$$

Then we have

$$\bar{x} = \frac{M_x}{m} = \frac{12 \text{ kgm}}{12\text{m}} = 1 \text{ m}$$

$$\bar{y} = \frac{M_y}{m} = \frac{4 \text{ kgm}}{12\text{m}} = 0.333 \text{ m}$$

The center of mass of the system is:  $(1, 0.333)\text{m}$ .

Source: "Moments and Centers of Mass" by LibreTexts, <https://eng.libretexts.org/@go/page/67237>

## 7.1.3 The Center of Mass of a Thin Uniform Rod (Calculus Method)

Quite often, when the finding of the position of the center of mass of a distribution of particles is called for, the distribution of particles is the set of particles making up a rigid body. The easiest rigid body for which to calculate the center of mass is the thin rod because it extends in only one dimension. (Here, we discuss an ideal thin rod. A

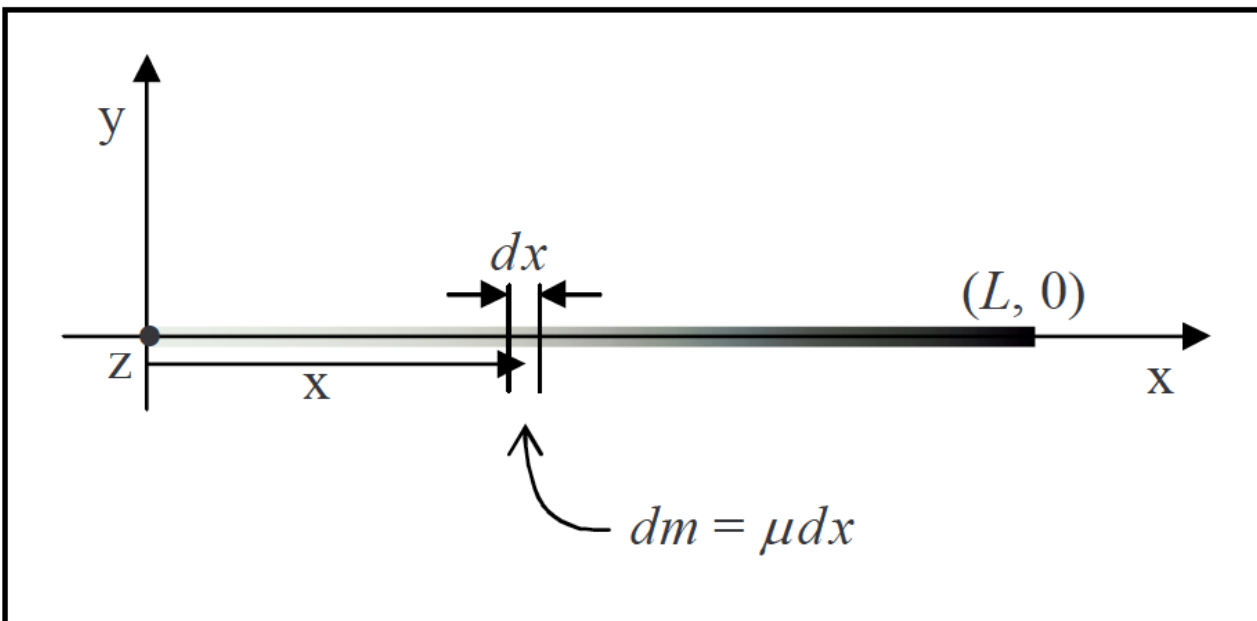
physical thin rod must have some nonzero diameter. The ideal thin rod, however, is a good approximation to the physical thin rod as long as the diameter of the rod is small compared to its length.)

In the simplest case, the calculation of the position of the center of mass is trivial. The simplest case involves a *uniform* thin rod. A uniform thin rod is one for which the linear mass density  $\mu$ , the mass-per-length of the rod, has one and the same value at all points on the rod. The center of mass of a uniform rod is at the center of the rod. So, for instance, the center of mass of a uniform rod that extends along the  $x$  axis from  $x = 0$  to  $x = L$  is at  $(L/2, 0)$ .

The *linear mass density*  $\mu$ , typically called *linear density* when the context is clear, is a measure of how closely packed the elementary particles making up the rod are. Where the linear density is high, the particles are close together.

To picture what is meant by a non-uniform rod, a rod whose linear density is a function of position, imagine a thin rod made of an alloy consisting of lead and aluminum. Further imagine that the percentage of lead in the rod varies smoothly from 0% at one end of the rod to 100% at the other. The linear density of such a rod would be a function of the position along the length of the rod. A one-millimeter segment of the rod at one position would have a different mass than that of a one-millimeter segment of the rod at a different position.

People with some exposure to calculus have an easier time understanding what linear density is than calculus-deprived individuals do because linear density is just the ratio of the amount of mass in a rod segment to the length of the segment, in the limit as the length of the segment goes to zero. Consider a rod that extends from 0 to  $L$  along the  $x$  axis. Now suppose that  $m_s(x)$  is the mass of that segment of the rod extending from 0 to  $x$  where  $x \geq 0$  but  $x < L$ . Then, the linear density of the rod at any point  $x$  along the rod, is just  $dm_s/dx$  evaluated at the value of  $x$  in question.



Source: Calculus-Based Physics 1, Jeffery W. Schnick. p143,  
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=ce74a181-ccde-491c-848d-05489ed182e7>

## 7.1.4 The Center of Mass of a Non-Uniform Rod

Now that you have a good idea of what we mean by linear mass density, we are going to illustrate how one determines the position of the center of mass of a non-uniform thin rod by means of an example.

### Example 3:

*Find the position of the center of mass of a thin rod that extends from 0 to 0.890 m along the  $x$  axis of a Cartesian coordinate system and has a linear density given by  $\mu = 0.650 \text{ kg/m}^3$*

In order to be able to determine the position of the center of mass of a rod with a given length and a given linear density as a function of position, you first need to be able to find the mass of such a rod. To do that, one might be tempted to use a method that works only for the special case of a uniform rod, namely, to try using  $m = \mu L$  with  $L$  being the length of the rod. The problem with this is, that  $\mu$  varies along the entire length of the rod. What value would one use for  $\mu$ ? One might be tempted to evaluate the given  $\mu$  at  $x = L$  and use that, but that would be acting as if the linear density were constant at  $\mu = \mu(L)$ . It is not. In fact, in the case at hand,  $\mu(L)$  is the maximum linear density of the rod, it only has that value at one point on the rod.

Instead, using integration, we find the equation:

$$m = \frac{bL^3}{3}$$

That can now be used to calculate the mass of a non-linear rod. The value of  $L$  is given as 0.890 m and we defined  $b$  to be the constant  $0.650 \text{ kg/m}^3$ , therefore

$$m = \frac{0.650 \frac{\text{kg}}{\text{m}^3} (0.890 \text{ m})^3}{3} = 0.1527 \text{ kg}$$

That's a value that will come in handy when we calculate the position of the center of mass.

Now, when we calculated the center of mass of a set of discrete particles (where a discrete particle is one that is by itself, as opposed, for instance, to being part of a rigid body) we just carried out a weighted sum in which each term was the position of a particle times its weighting factor and the weighting factor was that fraction, of the total mass, represented by the mass of the particle. We carry out a similar procedure for a continuous distribution of mass such as that which makes up the rod in question.

Once again, using integration, we find the equation:

$$\bar{x} = \frac{bL^4}{4m}$$

Now we substitute variables with values; the mass  $m$  of the rod that we found earlier, the constant  $b$  that we defined to simplify the appearance of the linear density function, and the given length  $L$  of the rod:

$$\bar{x} = \frac{\left( 0.650 \frac{\text{kg}}{\text{m}^3} \right) (0.890 \text{ m})^4}{4(0.1527 \text{ kg})} = 0.668 \text{ m}$$

This is our final answer for the position of the center of mass. Note that it is closer to the denser end of the rod, as we would expect.

Source: Calculus-Based Physics 1, Jeffery W. Schnick. p144,



<https://openlibrary.ecampusontario.ca/catalogue/item/?id=ce74a181-ccde-491c-848d-05489ed182e7>

### Key Takeaways

*Basically:* When there are multiple objects, the center of mass is the location in the x, y, and z directions between the objects.

*Application:* To calculate the acceleration or use  $F = ma$ ,  $m$  is the total mass at the center of mass.

*Looking Ahead:* The next section will look at how to calculate the center of mass for a complex object.

## 7.2 Center of Mass: Composite Shapes

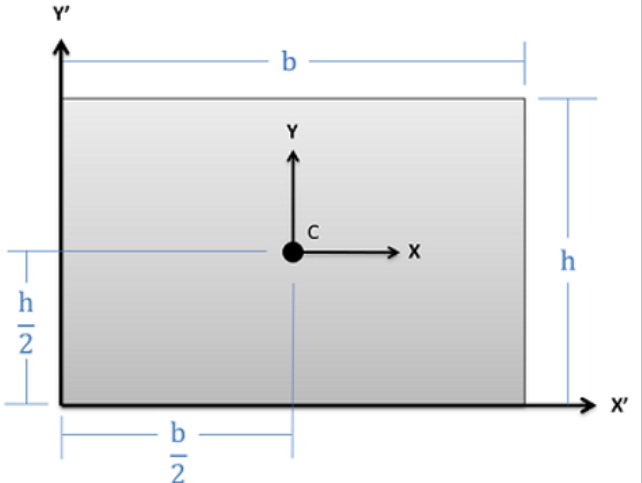
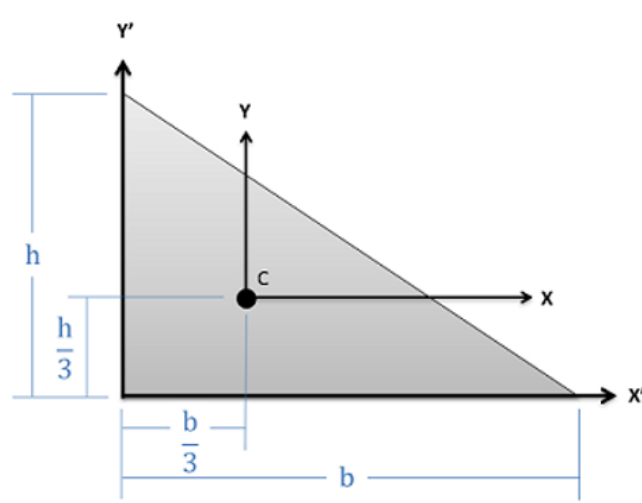
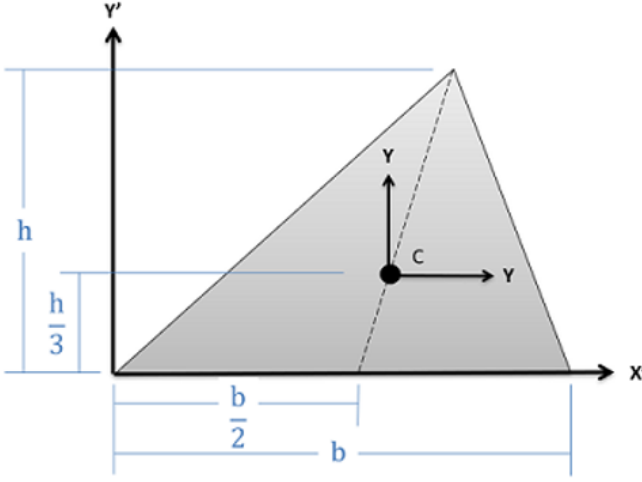
Instead of integrating to find the center of mass, we can split an object up into recognizable shapes to determine the center of mass. This is faster than the integration method, and allows for many different ways to find the answer.

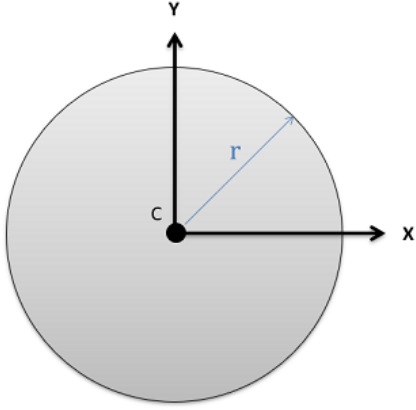
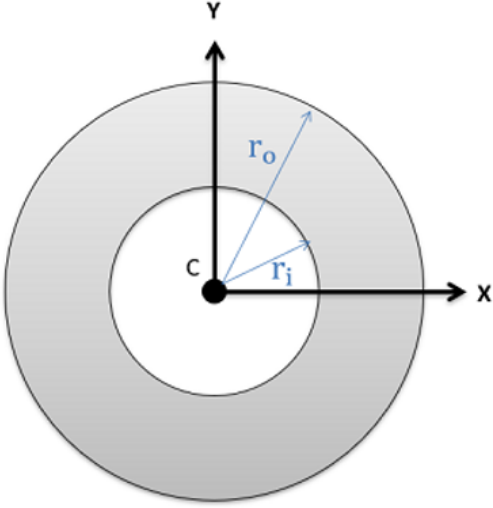
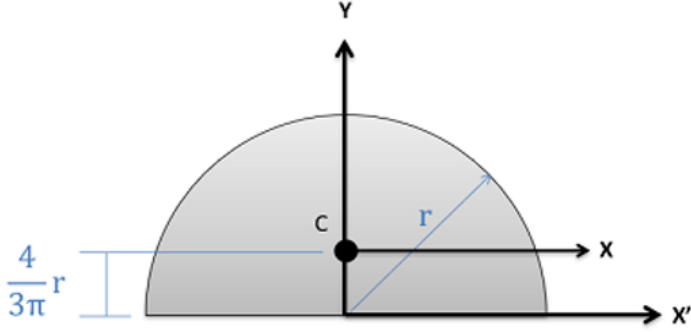
In this section, the terms 'centroid' and 'center of mass' are used, somewhat interchangeably. The centroid is the geometric center. The center of mass takes into account the density of an object. If the density is the same throughout an object, then the centroid is the same as the center of mass. For the rigid beams we will be looking at in statics, the centroid is at the same location as the center of mass.

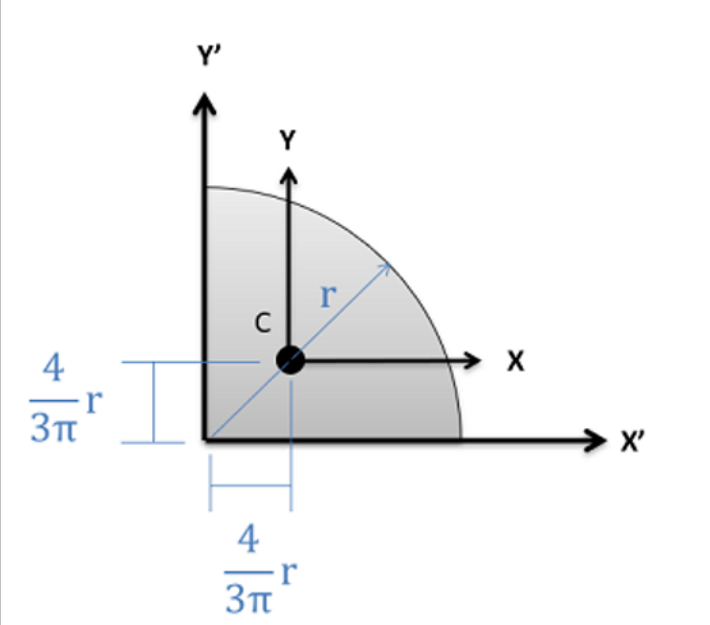
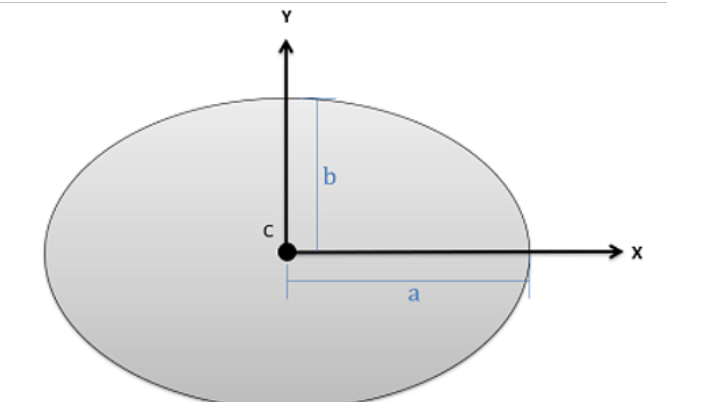
### 7.2.1 Centroid Tables

We use the centroid tables that are listed below to combine the locations of the centers of mass for each shape. Approximations are needed as real life objects are rarely perfectly square or circular, but if they are symmetric, it makes it easier to approximate.

The locations of the center of mass ( $r_{cm}$ ) are as follows. The source for the images are from Jacob Moore et al. <http://mechanicsmap.psu.edu/websites/centroidtables/centroids2D/centroids2D.html>

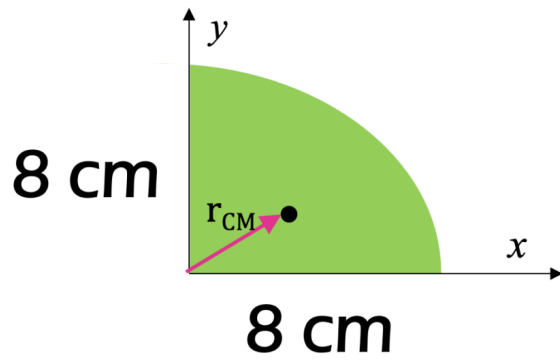
Shape	Images from Jacob Moore et al. <a href="http://mechanicsmap.psu.edu/websites/centroidtables/centroids2D/centroids2D.html">http://mechanicsmap.psu.edu/websites/centroidtables/centroids2D/centroids2D.html</a>	$r_{cm}$
<b>Rectangle</b>		$Area = bh$ $r_{cm} = \left( \frac{b}{2}, \frac{h}{2} \right)$
<b>Right Triangle</b>		$Area = \frac{1}{2}bh$ $r_{cm} = \left( \frac{b}{3}, \frac{h}{3} \right)$
<b>Triangle</b>		$Area = \frac{1}{2}bh$ $r_{cm} = \left( \frac{b}{2}, \frac{h}{3} \right)$

Circle		$Area = \pi r^2$ $r_{cm} = \left[0, 0\right]$
Circular Annulus		$Area = \pi (r_o^2 - r_i^2) \quad \text{\textit{r}_o = \text{outer radius} \text{\textit{r}_i = \text{inner radius}}}$ $r_{cm} = \left[0, 0\right]$
Semicircle		$Area = \frac{\pi}{2} r^2$ $r_{cm} = \left[0, \frac{4}{3\pi} r\right]$

<p><i>Quarter Circle</i></p>		$Area = \frac{\pi}{4} r^2$ $r_{cm} = \left( \frac{4}{3\pi} r, \frac{4}{3\pi} r \right)$
<p><i>Ellipse</i></p>		$Area = \pi a b$ $r_{cm} = (0, 0)$

### Example 1: A Single Object

Where is cm of:



To find the cm, select the appropriate shape from the above table.

.

.

.

.

.

Quarter Circle		$Area = \frac{\pi}{4} r^2$ $r_{cm} = \left[ \frac{4}{3\pi} r, \frac{4}{3\pi} r \right]$
----------------	--	---

Use the equation to solve for  $r_{cm}$ .

$$\mathbf{r}_{CM} = \begin{bmatrix} \frac{4}{3\pi}x \\ \frac{4}{3\pi}y \\ 0 \end{bmatrix}$$

$$\mathbf{r}_{CM} = \begin{bmatrix} \frac{4}{3\pi} * 8cm \\ \frac{4}{3\pi} * 8cm \\ 0 \end{bmatrix}$$

$$\mathbf{r}_{CM} = \begin{bmatrix} 3.4 \\ 3.4 \\ 0 \end{bmatrix} \text{ cm}$$

This is an appropriate answer as it is less than the midpoint, where it would be for a square:

$$\underline{\vec{r}}_{cm} = [3.4\text{cm}, 3.4\text{cm}, 0]$$

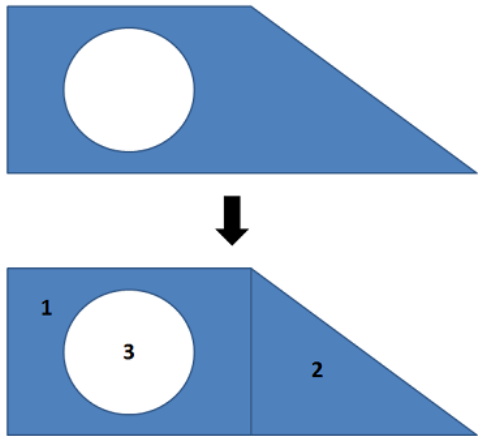
## 7.2.2 Composite Shapes

To find the center of mass of an object, you:

1. Define an origin
2. Split the object up into recognizable shapes
3. Find the center of mass (cm) of each shape from the origin
4. Calculate the mass of each part:  $\rho = \frac{m}{V}$  (To find the centroid, this step can be skipped and only the area or volume is used).
5. Use the weighted cm equations to find the x cm. Repeat for y and z.

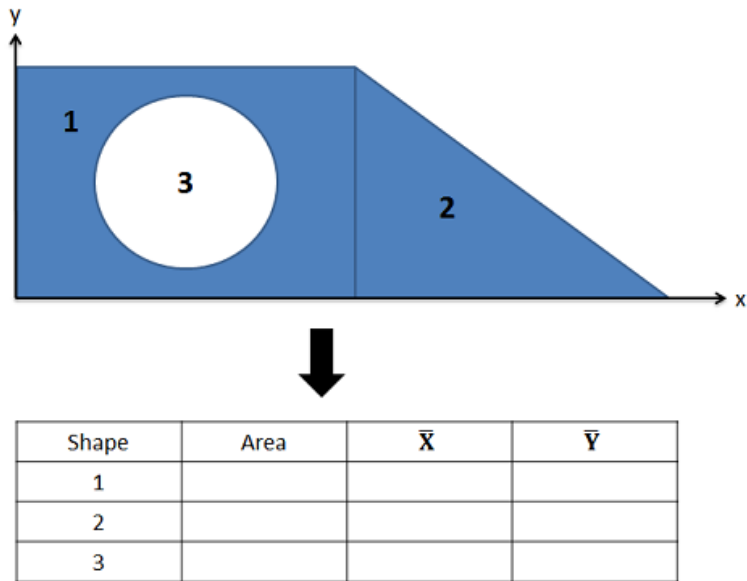
$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i} \quad \bar{y} = \frac{\sum m_i y_i}{\sum m_i} \quad \bar{z} = \frac{\sum m_i z_i}{\sum m_i}$$

Start the process by labeling an origin point and axes on your shape. It will be important to measure all locations from the same point. Next, we must break our complex shape down into several simpler shapes. This may include areas or volumes (which we will count as positive areas or volumes) or holes (which we will count as negative areas or volumes). Each of these shapes will have a centroid (□) or center of mass (□) listed on the diagram.



For the shape shown at the top, we can break it down into a rectangle (1), a right triangle (2), and a circular hole (3). Each of these simple shapes is something we have listed in the centroid table to the right.

Once we have identified the different parts, we will create a table listing the area or volume of each piece, and the  $\bar{x}$  and  $\bar{y}$  centroid coordinates (or  $\bar{x}$ ,  $\bar{y}$ , and  $\bar{z}$  coordinates in 3D). It is important to remember that each coordinate you list should be relative to the same base origin point that you drew in earlier. You may need to mentally adjust diagrams in the centroid tables so that the shape is oriented in the right direction, and account for the placement of the shape relative to the axes in your diagram.



For each of the shapes, we need to find the area and the x and y coordinates of the centroid. Remember to find the centroid coordinates relative to a single set of axes that is the same for all shapes.

Once you have the areas and centroid coordinates for each shape relative to your origin point, you can find the  $\bar{x}$  and  $\bar{y}$  coordinate of the centroid for the overall shape with the following formulas. Remember that areas or volumes for any shape that is a hole or cutout in the design will be a negative area in your formula.

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i} \quad \bar{y} = \frac{\sum_{i=1}^n m_i y_i}{\sum_{i=1}^n m_i}$$



This generalized formula to find the centroid's  $\bar{x}$ -location is simply Area 1 times  $\bar{x}_1$  plus Area 2 times  $\bar{x}_2$ , plus Area 3 times  $\bar{x}_3$ , adding up as many shapes as you have in this fashion and then dividing by the overall area of your combined shape. The equations are the same for the  $\bar{y}$ -location of the overall centroid, except you will instead be using  $\bar{y}$ -values in your equations.

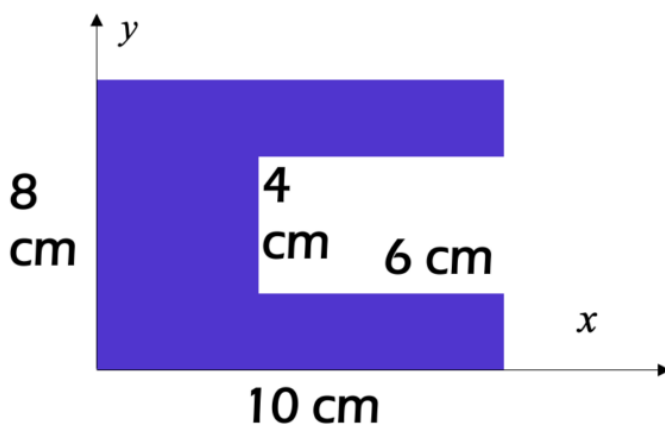
For centroids in three dimensions we will simply use volumes in place of areas, and we will have a  $\bar{z}$  coordinate for our centroid as well as the  $\bar{x}$  and  $\bar{y}$  coordinates.

Source: Jacob Moore et al.

[http://mechanicsmap.psu.edu/websites/A2\\_moment\\_integrals/method\\_of\\_composite\\_parts/methodofcompositeparts.html](http://mechanicsmap.psu.edu/websites/A2_moment_integrals/method_of_composite_parts/methodofcompositeparts.html)

## Example 2 - A Composite Object

For the following C shape, where is the center of mass?

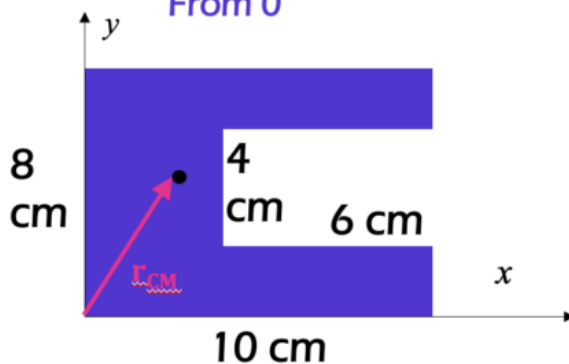


Depth = 1 cm  
Density = 5 kg/cm<sup>3</sup>

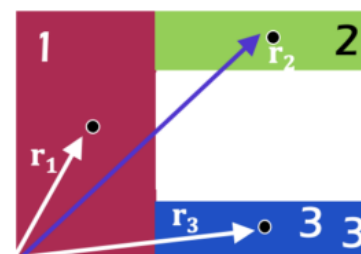
What is cm?

1. Pick a point (0, midpoint,  $m_1$ ,  $m_2$  etc)

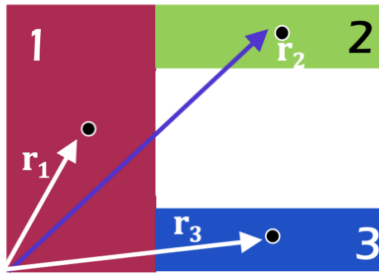
From 0



2. Separate into smaller shapes



## 3. Measure distance to centroid of each shape from point



$$r_1 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \text{ cm} \quad r_2 = \begin{bmatrix} 7 \text{ cm} \\ 7 \text{ cm} \\ 0 \end{bmatrix}$$

$$r_3 = \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} \text{ cm}$$

## 4. Calculate masses

$$\text{Density } \rho = 5 \text{ kg/cm}^3$$

$$m = \rho V = \rho A z = (5 \text{ kg/cm}^3) * (1 \text{ cm}) * A \quad m = 5 \text{ kg/cm}^2 * A$$

$$m_1 = 5 \text{ kg/cm}^2 * 4 \text{ cm} * 8 \text{ cm} = 160 \text{ kg}$$

$$m_2 = 5 \text{ kg/cm}^2 * 6 \text{ cm} * 2 \text{ cm} = 60 \text{ kg}$$

$$m_3 = m_2 = 60 \text{ kg}$$

$$m_{\text{tot}} = 160 + 60 + 60 \text{ kg} = 280 \text{ kg}$$

$$m_1 = 160 \text{ kg} \quad m_2 = 60 \text{ kg} \quad m_3 = 60 \text{ kg} \quad m_{\text{tot}} = 280 \text{ kg}$$

5. Calculate  $x_{cm}$ 

$$x_{CM} = \frac{\sum m_i x_i}{\sum m} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_{tot}}$$

$$x_{CM} = \frac{160kg * 2cm + 60kg * 7cm + 60kg * 7cm}{280 kg} = \frac{1160kg * cm}{280 kg} = 4.14 cm$$

6. Repeat for  $y_{cm}$ 

$$y_{CM} = \frac{\sum m_i y_i}{\sum m} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_{tot}}$$

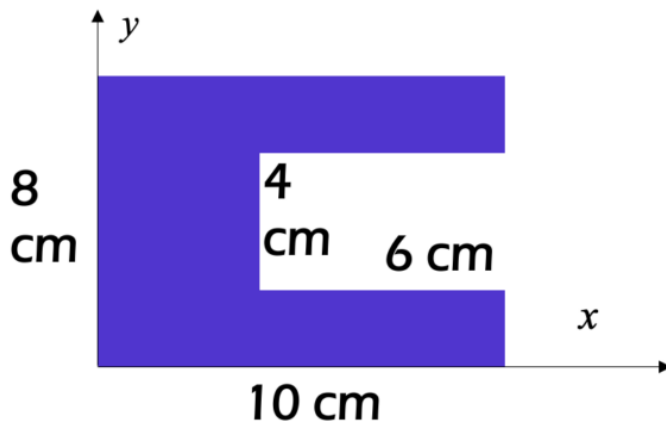
$$y_{CM} = \frac{160kg * 4cm + 60kg * 7cm + 60kg * 1cm}{280 kg} = \frac{1120kg * cm}{280 kg} = 4 cm$$

$$r_{cm} = \begin{bmatrix} 4.14 \\ 4 \\ 0 \end{bmatrix} cm$$

This example used the density to calculate mass. To find the centroid, the area could have been used instead of the mass. However to find the center of mass, the density was used to calculate the mass of each object individually:  $\rho = \frac{m}{V}$ .

*Example 3: Using Subtraction instead of Addition*

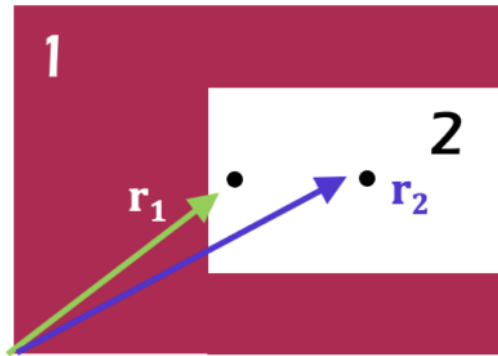
For the following C shape, where is the center of mass? (Use different shapes than above).



Depth = 1 cm  
Density = 5 kg/cm<sup>3</sup>

What is cm?

## SUBTRACTION



$$r_1 = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} \text{ cm}$$

$$r_2 = \begin{bmatrix} 7 \text{ cm} \\ 4 \text{ cm} \\ 0 \end{bmatrix}$$

$$m_1 = 80 \text{ cm}^2 \cdot 5 \text{ kg/cm}^2 = 400 \text{ kg}$$

$$m_2 = 24 \text{ cm}^2 \cdot 5 \text{ kg/cm}^2 = 120 \text{ kg}$$

$$m_{\text{tot}} = 400 - 120 \text{ kg} = 280 \text{ kg}$$

$$x_{\text{CM}} = \frac{m_1 x_1 - m_2 x_2}{m_{\text{tot}}}$$

$$x_{\text{CM}} = \frac{400 \text{ kg} \cdot 5 \text{ cm} - 120 \text{ kg} \cdot 7 \text{ cm}}{280 \text{ kg}} = \frac{1160 \text{ kg} \cdot \text{cm}}{280 \text{ kg}} = 4.14 \text{ cm}$$

$$y_{\text{CM}} = \frac{m_1 y_1 - m_2 y_2}{m_{\text{tot}}}$$

$$y_{\text{CM}} = \frac{400 \text{ kg} \cdot 4 \text{ cm} - 120 \text{ kg} \cdot 4 \text{ cm}}{280 \text{ kg}} = \frac{1120 \text{ kg} \cdot \text{cm}}{280 \text{ kg}} = 4 \text{ cm}$$

$$r_{\text{cm}} = \begin{bmatrix} 4.14 \\ 4 \\ 0 \end{bmatrix} \text{ cm}$$

The same answer is reached! This method involved only 2 shapes instead of 3.

There are more examples at [http://mechanicsmap.psu.edu/websites/A2\\_moment\\_intergrals/method\\_of\\_composite\\_parts/methodofcompositeparts.html](http://mechanicsmap.psu.edu/websites/A2_moment_intergrals/method_of_composite_parts/methodofcompositeparts.html) with pdfs and video solutions.

#### Key Takeaways

*Basically:* When there are complicated shapes, the center of mass can be found by breaking the shape up into better known shapes.

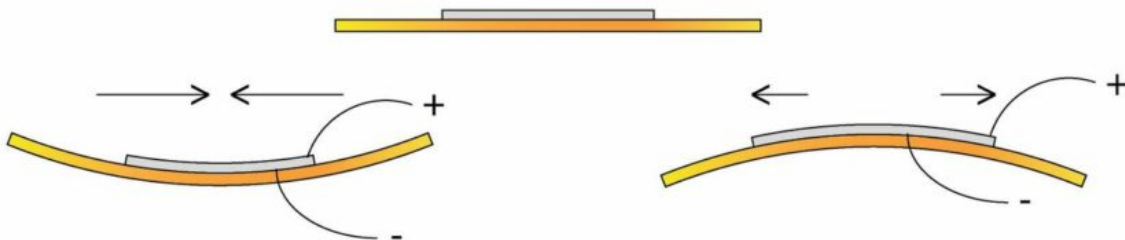
*Application:* To calculate the inertia of an object rotating about its center of mass, you will need to know where the center of mass is.

*Looking Ahead:* The next section will look at types of inertia.

## 7.3 Types of Inertia

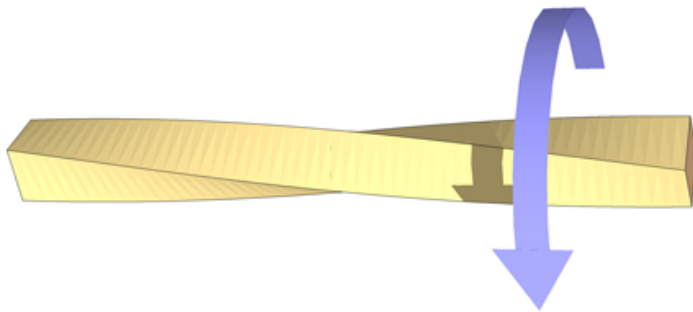
There are multiple kinds of inertia. In this course, when we talk about inertia, we usually refer to mass moment of inertia. For this course, you need to know the names of three types of moment of inertia (MOI), when to use each, and the units for each.

Area moment of inertia is used in structures to determine how stiff a beam is, or how much it will deflect. The unit is  $\text{m}^4$  or  $\text{ft}^4$ .



Source: Sonitron Support [https://commons.wikimedia.org/wiki/File:Piezo\\_bending\\_principle.jpg](https://commons.wikimedia.org/wiki/File:Piezo_bending_principle.jpg)

Product moment of inertia is a shaft's resistance to torsion (or twisting). The unit is  $\text{m}^4$  or  $\text{ft}^4$ .



Source: Orion8  
[https://commons.wikimedia.org/wiki/File:Twisted\\_bar.png](https://commons.wikimedia.org/wiki/File:Twisted_bar.png)

Mass moment of inertia is the rotational equivalent to mass. A really heavy object is hard to move, hence it resists motion. An object with a really big inertia is hard to rotate, hence it resists rotation. An inertia is a quantity of how mass is distributed around a body, such as an ice skater spinning with their arms in or spread out. The unit is  $\text{kgm}^2$  or  $\text{slugft}^2$ . We will talk more about mass moment of inertia in the next sections.



Source: K. "bird"  
[https://commons.wikimedia.org/wiki/File:Elena\\_Sokolova\\_04\\_NHK\\_2.jpg](https://commons.wikimedia.org/wiki/File:Elena_Sokolova_04_NHK_2.jpg)



Source: Sandro Halank  
[https://commons.wikimedia.org/wiki/File:2020-01-11\\_Women%27s\\_Single\\_Figure\\_Skating\\_Short\\_Program\\_\(2020\\_Winter\\_Youth\\_Olympics\)\\_by\\_Sandro\\_Halank%E2%80%9393668.jpg](https://commons.wikimedia.org/wiki/File:2020-01-11_Women%27s_Single_Figure_Skating_Short_Program_(2020_Winter_Youth_Olympics)_by_Sandro_Halank%E2%80%9393668.jpg)

Here is a table summarizing the three types of inertia:

Type of Inertia	Mass MOI	Area MOI	Product MOI
<b>Definition</b>	Resistance to rotation	Symmetric bending	Asymmetric bending
<b>Equation</b>	$I = \int_m r^2 dm$	$I_x = \int_A y^2 dA$ $I_y = \int_A x^2 dA$	$I_{xy} = \int_A xy^2 dA$
<b>Unit</b>	kgm <sup>2</sup> & slug ft <sup>2</sup>	m <sup>4</sup> & ft <sup>4</sup> or in <sup>4</sup>	m <sup>4</sup> & ft <sup>4</sup> or in <sup>4</sup>
<b>Use</b>	An object's resistance to rotating like an ice skater twirling	A beam's resistance to bending	A shaft's resistance to torsion (twisting)
	Dynamics $\sum M = I\alpha$	Used for flexural stress and deflection of beams, column buckling	

A **moment integral**, as the name implies, is the general concept using integration to determine the net moment of a force that is spread over an area or volume. Because moments are generally a force times a distance, and because distributed forces are spread out over a range of distances, we will need to use calculus to determine the net moment exerted by a distributed force.

$$\int M = \int f(d) \cdot d$$

Beyond the most literal definition of a moment integral, the term 'moment integral' is also general applied the process of integrating distributed **areas** or **masses** that will be resiting some moment about a set axis.

Some of the applications of moment integrals include:

1. Finding point loads that are equivalent to distributed loads (the **equivalent point load**)
2. Finding the **centroid** (geometric center) or **center of mass** for 2D and 3D shapes.
3. Finding the **area moment of inertia** for a beam cross section, which will be one factor in that beam's resistance to bending.
4. Finding the **polar area moment of inertia** for a shaft cross section, which will be one factor in that shaft's resistance to torsion.
5. Finding the **mass moment of inertia**, indicating a body's resistance to angular accelerations.

When looking at moment integrals, there are number of different types of moment integrals. These will include moment integrals in one dimension, two dimensions, and three dimensions, moment integrals of force functions, of areas/volumes, or of mass distributions, first order or second order moment integrals, and rectangular or polar moment integrals.

Any combination of these different types is possible (for example a first, rectangular, 2D, area moment integral or



a second, polar, 3D, mass moment integral). However, only some combinations will have practical applications and will be discussed in detail on future pages.

## *1D, 2D, and 3D Moment Integrals*

Technically we can take the moment integral in any number of dimensions, but for practical purposes we will never deal with moment integrals beyond 3 dimensions. The number of dimensions will affect the complexity of the calculations (with 3D Moment integrals being the more involved than 1D or 2D moment integrals), but the nature of the problem will dictate the dimensions needed. Often this is not listed in the type of moment integral, requiring you to assume the type based on the context of the problem.

## *Force, Area/Volume, and Mass Moments Integrals*

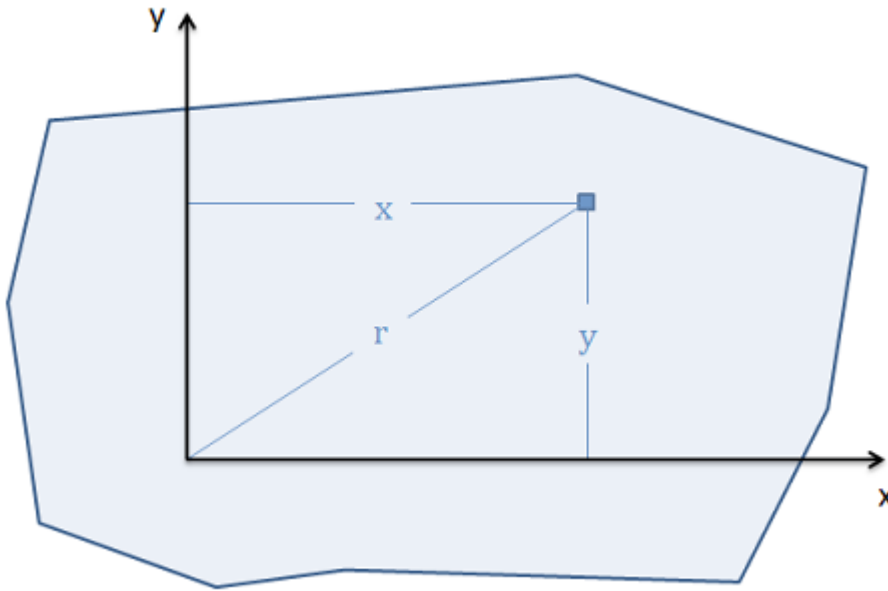
The next distinction in moment integrals is regard what we are integrating. Generally, we can integrate force functions over some distance, area, or volume, we can integrate the area or volume function itself, or we can integrate the mass distribution over the area or volume. Each of these types of moment integrals has a different purpose and will start with a different mathematical function to integrate, but the integration process beyond that will be very similar.

## *First vs. Second Moments Integrals*

For moment integrals we will always be multiplying the force function, area or volume function, of the mass distribution function by a distance, or a distance squared. First moment integrals just multiply the initial function by the distance, while second moment integrals multiply the function by the distance squared. Again the type of moment integral we will use depends upon our application, with things like equivalent point load, centroids, and center of mass relying on first moment integrals, and area moments of inertia, polar moments of inertia, and mass moments of inertia relying on second moment integrals. As you can probably deduce from this list, second moment integrals, are often labeled as a '**moment of inertia**'

## *Rectangular vs. Polar Moments Integrals*

Finally we will talk about **rectangular moments integrals** versus **polar moments integrals**. This is a difference in how we define the distance in our moment integral. Let's start with the distinction in 2D. If our distance is measured from some axis (for example the x-axis, or the y-axis) then it is a rectangular moment integral. If on the other hand the distance is measured from some point (such as the origin) then it is a polar moment integral.



*In 2D, if we measure the distance from some axis (similar to what  $x$  and  $y$  do here) then we have a rectangular moment integral. If we measure the distance from some point (such as  $r$  does) then we have a polar moment integral.*

This distinction is important for how we will take the integral. For rectangular moment integrals we will move left to right or bottom to top. For polar moment integrals we will instead take the integral by radiating out from the center point.

In three dimensional problems, the definitions change slightly. For rectangular moment integrals the distance will be measured from some plane (such as the  $xy$  plane,  $xz$  plane, or  $yz$  plane). Again we will integrate left to right, bottom to top, or now back to front with distances corresponding to the  $x$ ,  $y$  or  $z$  coordinates of that point. For a polar moment integrals the distance will be measured from some axis (such as the  $x$ ,  $y$ , or  $z$  axis), and we will integrate by radiating outward from that axis.

Source: Jacob Moore et al.

[http://mechanicsmap.psu.edu/websites/A2\\_moment\\_integrals/moment\\_integrals/momentintegrals.html](http://mechanicsmap.psu.edu/websites/A2_moment_integrals/moment_integrals/momentintegrals.html)

### Key Takeaways

*Basically:* There are many types of inertia, including area, product, and mass. They have different units and represent different physical quantities of an object.

*Application:* A beam's resistance to bending (area moment of inertia), a shaft's resistant to torsion (product/polar moment of inertia), and an object's resistance to rotating (mass moment of inertia).

*Looking Ahead:* Area and product moment of inertia will be covered more in structures. Mass moment of inertia is used in statics and dynamics. The next section will look at how to calculate the mass moment of inertia.

# 7.4 Mass Moment of Inertia

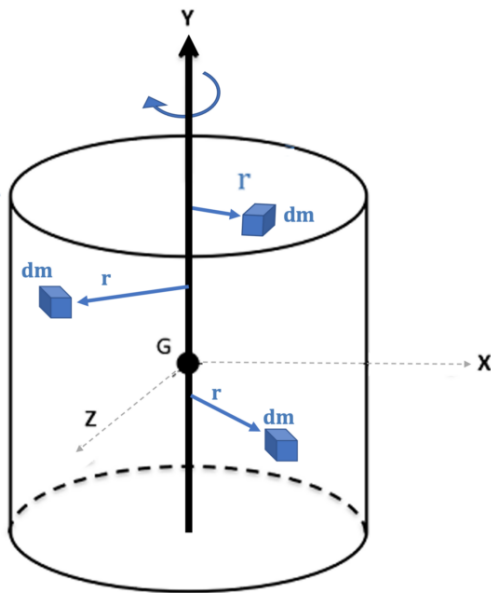
## 7.4.1 Intro to Mass Moment of Inertia

Mass moment of inertia, or inertia as it will be referred to from here on, is resistance to rotation. The bigger the inertia, the slower the rotation.  $\sum M = I\alpha$ . Inertia is always positive and has units of  $\text{kgm}^2$  or  $\text{slugft}^2$ .

For an infinitesimal unit of mass, the inertia depends on how far it is from the axis of rotation.

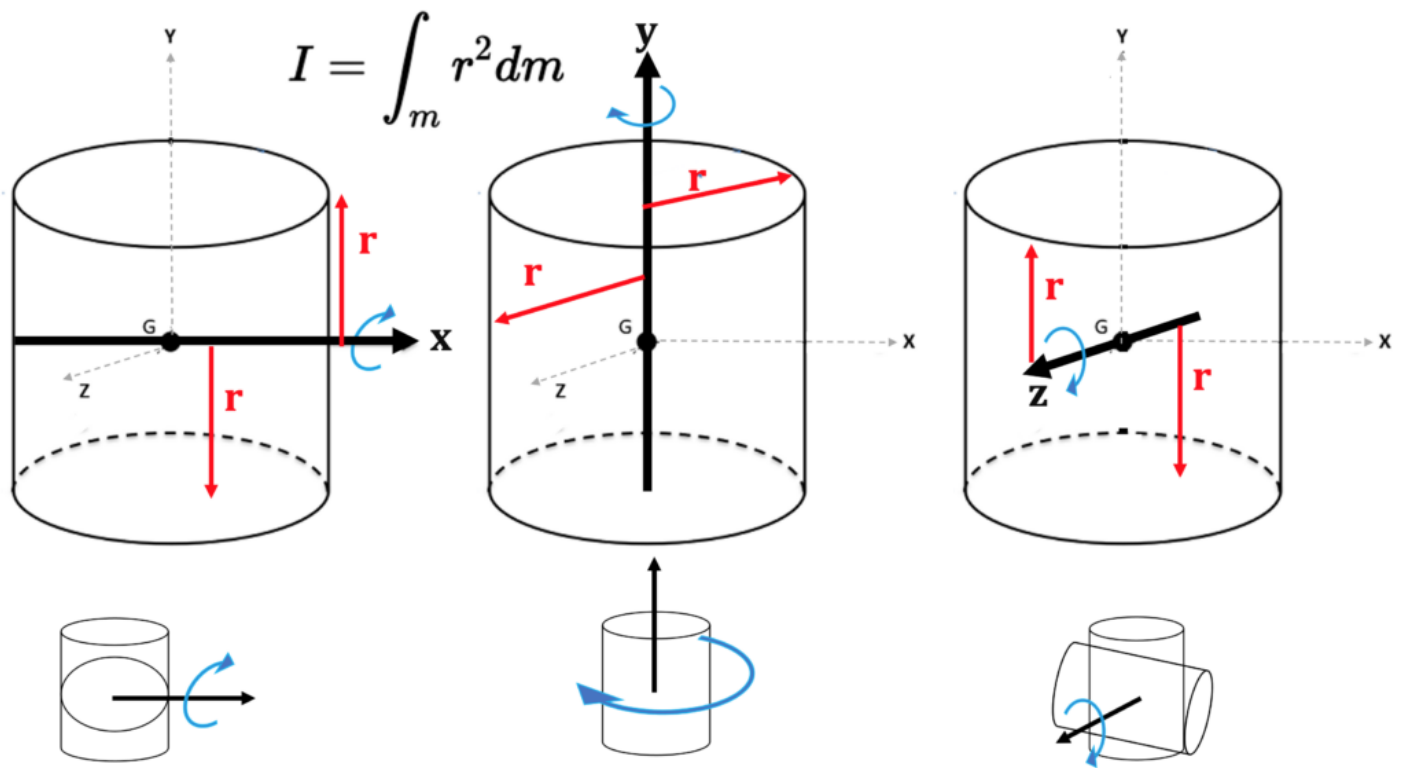
$$I = \int_m r^2 dm$$

As shown in this image, each little  $dm$  at  $r$  distance from the axis of rotation ( $y$ ) is added up (through integration). If  $r$  is bigger, the inertia is bigger.



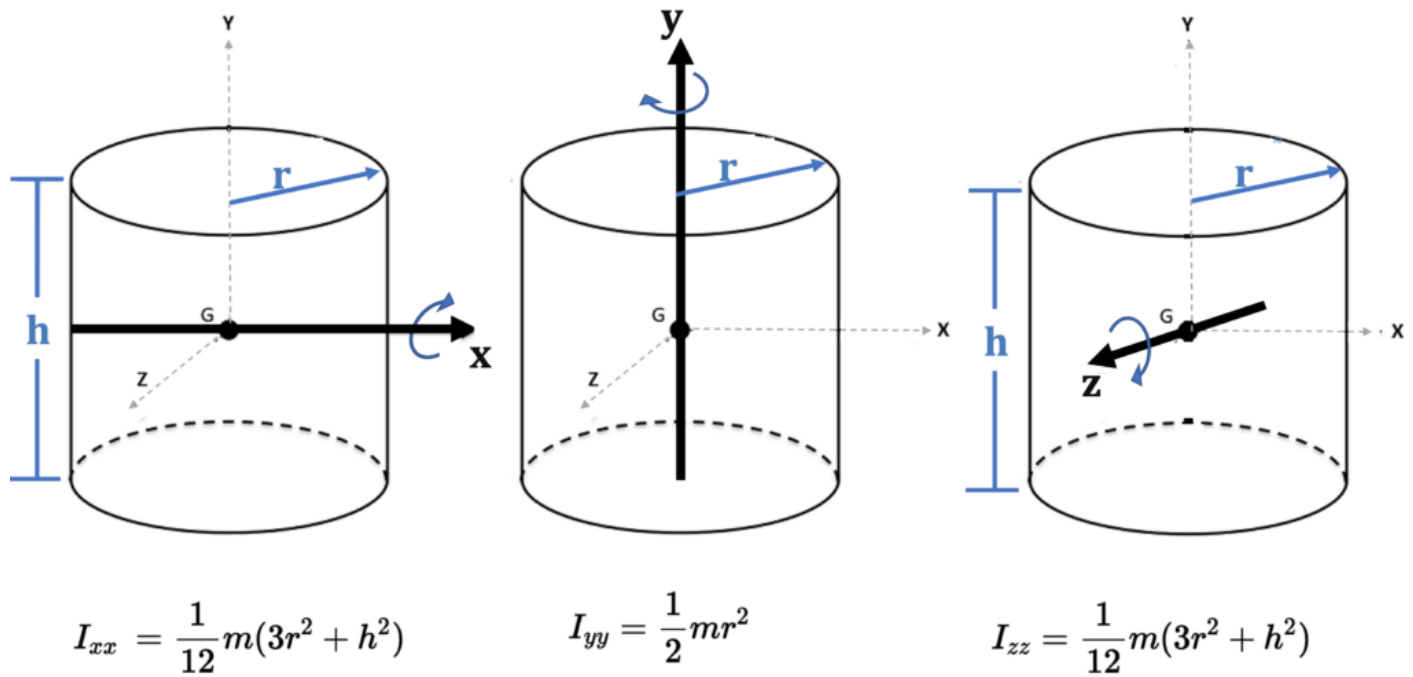
$$I = \int_m r^2 dm$$

If there is more mass closer to the axis of rotation, the inertia is smaller. A skill that you can develop is your visualization of the rotation about each axis. As shown in the following figure, rotating about the different axes will produce different types of rotation. You can imagine sticking your pencil into an object and twisting along that axis. In this image, rotation about the  $y$  axis and  $x$  axis produce different types of rotation. Due to the symmetry, rotation about the  $x$  axis and  $z$  axis looks identical.



The red  $r$ 's in this image show the distance that is being measured when adding up each little infinitesimal  $dm$ . Notice how the  $r$  changes direction from  $x$  to  $y$  but looks the same between  $x$  and  $z$ .

Equations have been developed for common shapes so that you don't have to integrate every time you want to find the inertia of an object. The result is different for each axis, as shown in the following figure.

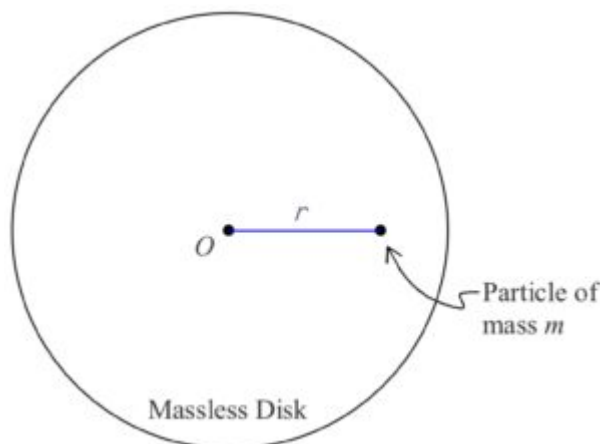


' $I_{xx}$ ' can be read as 'the inertia if rotating about the x-axis'. Notice for  $I_{xx}$  and  $I_{zz}$  that the height and radius of the cylinder affect the inertia, whereas for  $I_{yy}$ , only the radius is considered.

The equations for each of the objects are listed in a table below. First is a second explanation of inertia.

We start by constructing, in our minds, an idealized object for which the mass is all concentrated at a single location which is not on the axis of rotation: Imagine a massless disk rotating about an axis through the center of the disk and perpendicular to its faces.

Let there be a particle of mass  $m$  embedded in the disk at a distance  $r$  from the axis of rotation. Here's what it looks like from a viewpoint on the axis of rotation, some distance away from the disk:



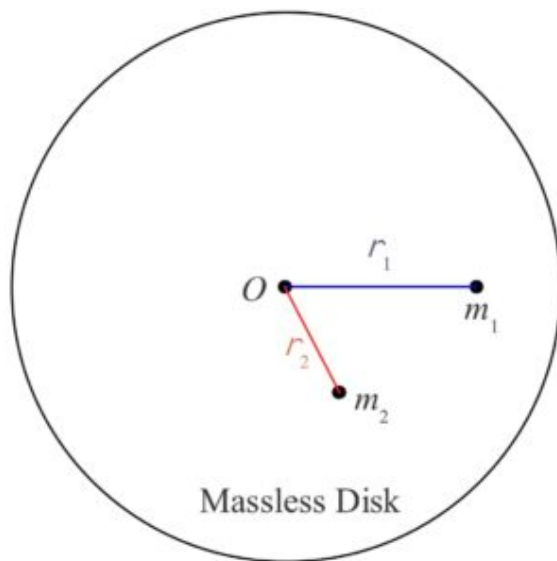
where the axis of rotation is marked with an  $O$ . Because the disk is massless, we call the moment of inertia of the

construction, the moment of inertia of a particle, with respect to rotation about an axis from which the particle is a distance  $r$ .

$$I = mr^2$$

is our equation for the moment of inertia of a particle of mass  $m$ , with respect to an axis of rotation from which the particle is a distance  $r$ .

Now suppose we have two particles embedded in our massless disk, one of mass  $m_1$  at a distance  $r_1$  from the axis of rotation and another of mass  $m_2$  at a distance  $r_2$  from the axis of rotation.



The moment of inertia of the first one by itself would be

$$I_1 = m_1 r_1^2$$

and the moment of inertia of the second particle by itself would be

$$I_2 = m_2 r_2^2$$

The total moment of inertia of the two particles embedded in the massless disk is simply the sum of the two individual moments of inertia.

$$I = I_1 + I_2$$

$$I = m_1 r_1^2 + m_2 r_2^2$$

This concept can be extended to include any number of particles. For each additional particle, one simply includes another  $m_i r_i^2$  term in the sum where  $m_i$  is the mass of the additional particle and  $r_i$  is the distance that the additional particle is from the axis of rotation. In the case of a rigid object, we subdivide the object up into an infinite set of infinitesimal mass elements  $dm$ . Each mass element contributes an amount of moment of inertia

$$dI = r^2 dm$$

to the moment of inertia of the object, where  $r$  is the distance that the particular mass element is from the axis of rotation.

Source: Calculus-Based Physics 1, Jeffery W. Schnick.

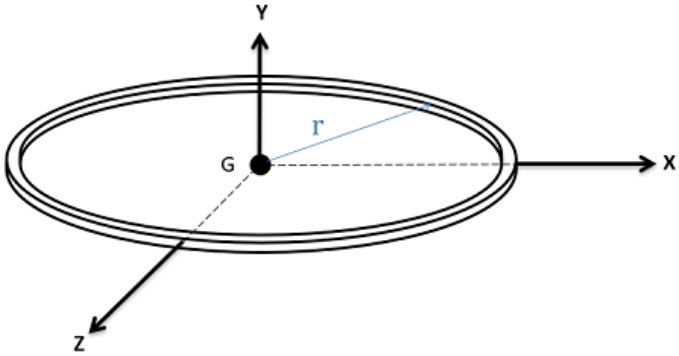
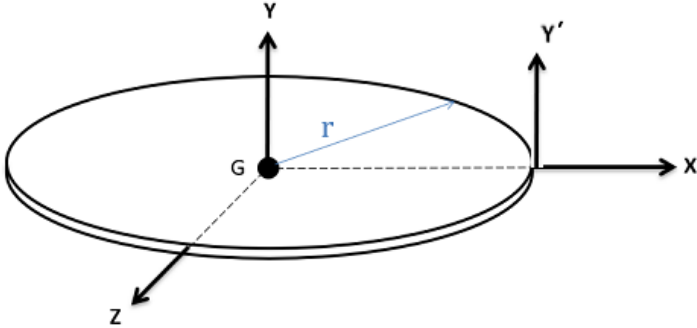
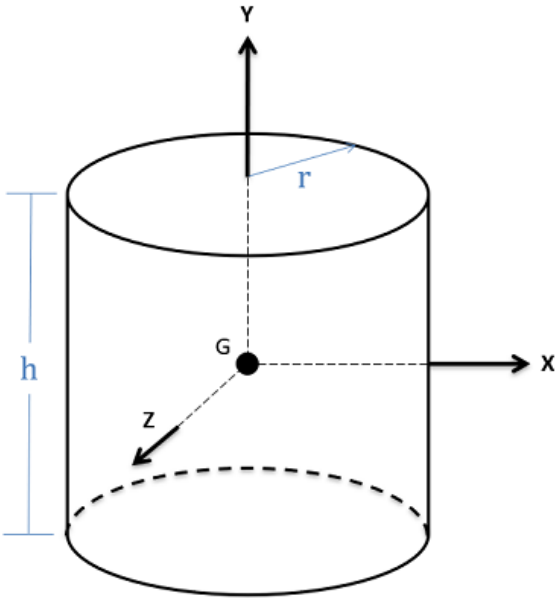
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=ce74a181-ccde-491c-848d-05489ed182e7>

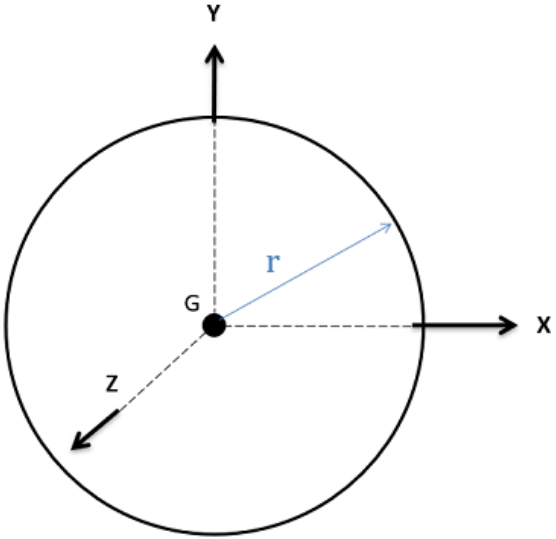
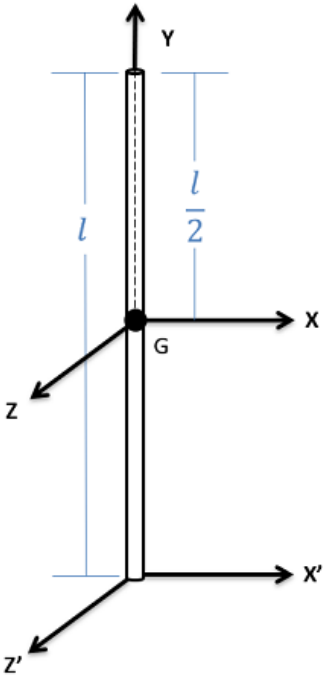
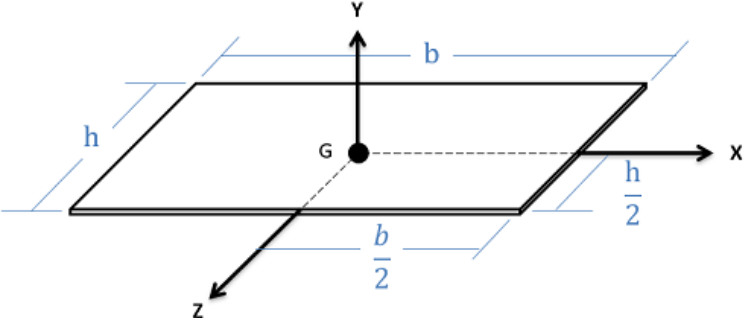
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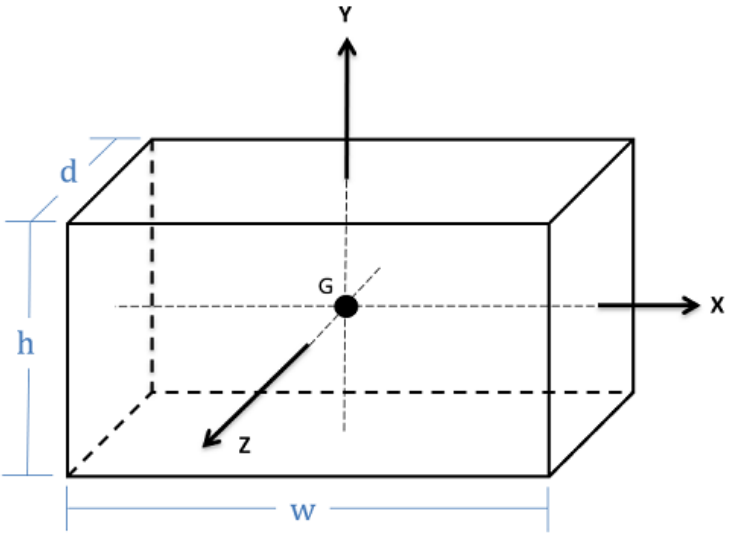
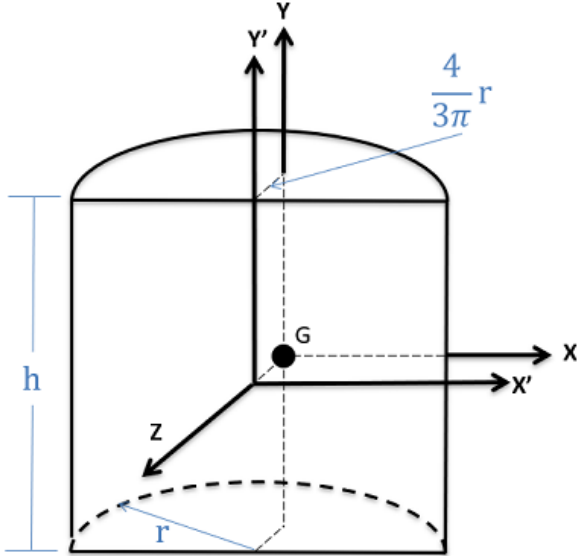
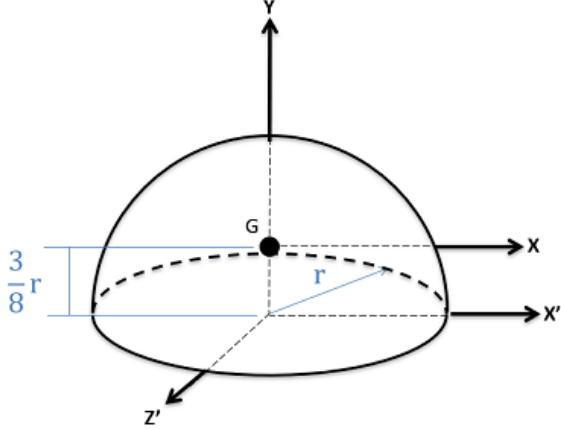
## 7.4.2 Inertia Table of Common Shapes

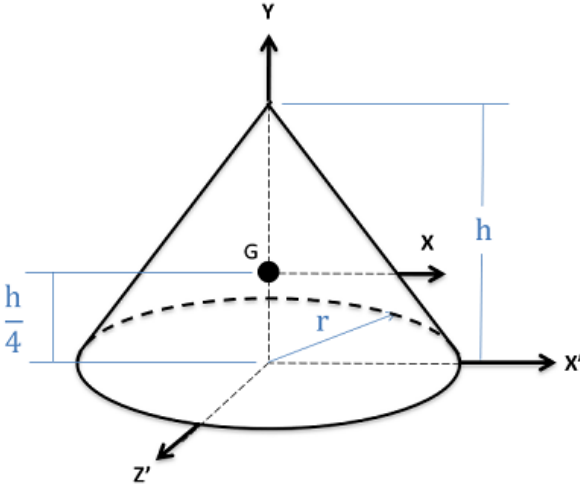
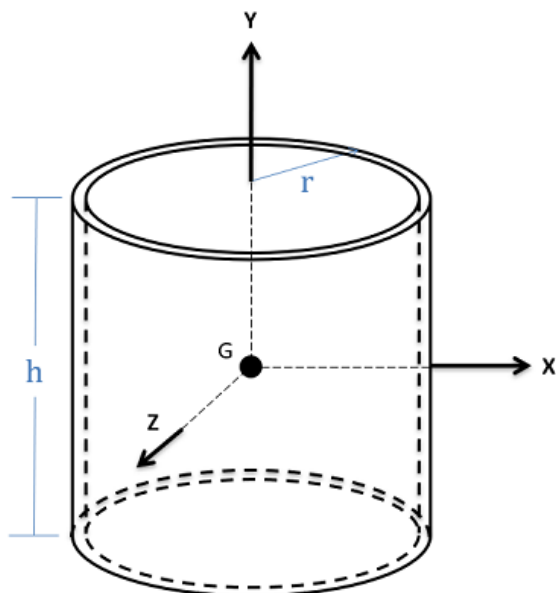
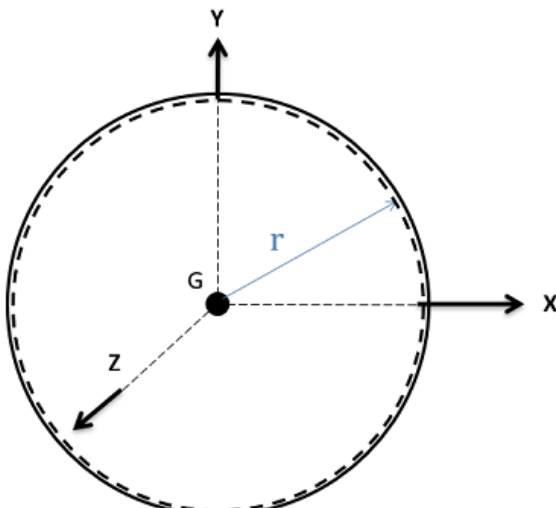
Specific inertia equations depending on the shape of the object and axis of rotation can be found below. Notice some of the shapes have multiple sets of axes:  $I_{xx}$  and  $I_{xx}'$ . There are multiple equations.

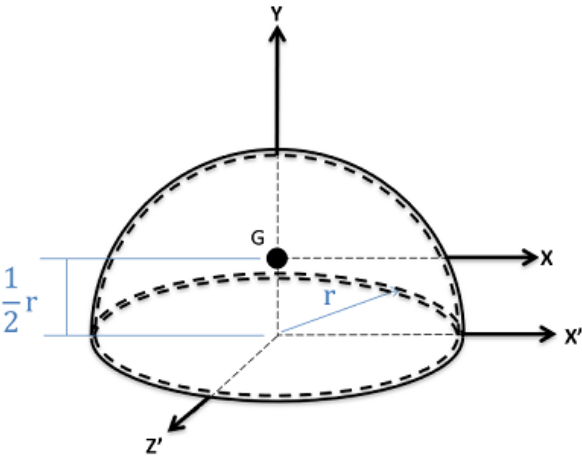


Symmetric Shapes		
<b>Thin Ring</b>		$I_{xx} = \frac{1}{2}mr^2 \quad I_{yy} = mr^2$ $I_{zz} = \frac{1}{2}mr^2$ $* \text{thickness} \ll 1$
<b>Circular Plate</b>		$I_{xx} = \frac{1}{4}mr^2$ $I_{yy} = \frac{1}{2}mr^2 \quad I_{zz} = \frac{1}{4}mr^2$ $I_{yy'} = \frac{3}{2}mr^2$ $* \text{thickness} \ll 1$
<b>Cylinder</b>		$I_{xx} = \frac{1}{12}m(3r^2 + h^2)$ $I_{yy} = \frac{1}{2}mr^2 \quad I_{zz} = \frac{1}{12}m(3r^2 + h^2)$ $\text{Volume} = \pi r^2 h$

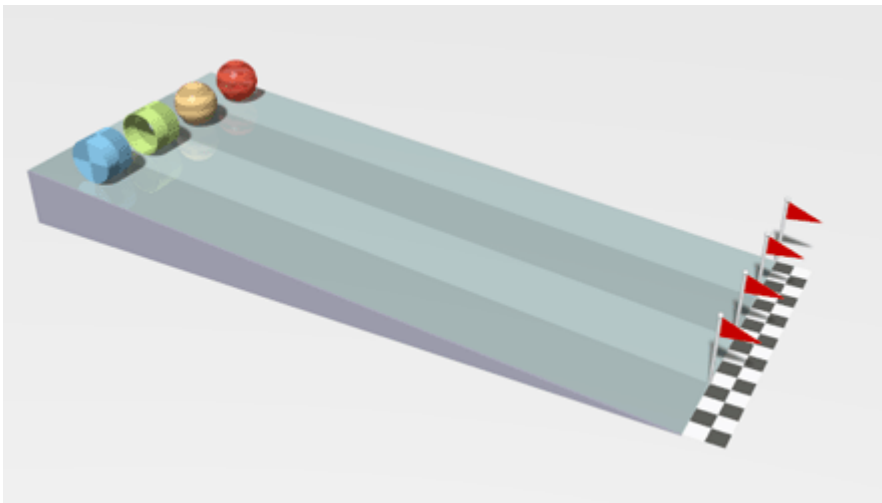
Sphere		$I_{xx} = \frac{2}{5}mr^2$ $I_{yy} = \frac{2}{5}mr^2$ $I_{zz} = \frac{2}{5}mr^2$ $Volume = \frac{4}{3}\pi r^3$
Slender Rod		$I_{xx} = \frac{1}{12}ml^2$ $I_{yy} = 0, I_{zz} = \frac{1}{12}ml^2$ $I_{xx}' = \frac{1}{3}ml^2$ $I_{zz}' = \frac{1}{3}ml^2$ <p>* radius <math>\ll</math> length</p>
Rectangular Plate		$I_{xx} = \frac{1}{12}mh^2$ $I_{yy} = \frac{1}{12}m(h^2 + b^2)$ $I_{zz} = \frac{1}{12}mb^2$ <p>* thickness <math>\ll 1</math></p>

<b>Rectangular Block</b>		$I_{xx} = \frac{1}{12}m(h^2 + d^2)$ $I_{yy} = \frac{1}{12}m(d^2 + w^2) \quad I_{zz} = \frac{1}{12}m(h^2 + w^2)$ $\text{Volume} = bwh$
Asymmetric Shapes		
<b>Half Cylinder</b>		$I_{xx} = \left( \frac{1}{4} - \frac{16}{9\pi^2} \right) mr^2 + \frac{1}{12}mh^2$ $I_{yy} = \left( \frac{1}{2} - \frac{16}{9\pi^2} \right) mr^2$ $I_{zz} = \left( \frac{1}{4} - \frac{16}{9\pi^2} \right) mr^2 + \frac{1}{12}mh^2$ $I_{xx'} = \frac{1}{12}m(3r^2 + h^2)$ $I_{yy'} = \frac{1}{2}mr^2$ $I_{zz'} = \frac{1}{12}m(3r^2 + h^2)$ $\text{Volume} = \frac{1}{2}\pi r^2 h$
<b>Hemisphere</b>		$I_{xx} = \frac{83}{320}mr^2$ $I_{yy} = \frac{2}{5}mr^2$ $I_{zz} = \frac{83}{320}mr^2$ $I_{xx'} = \frac{2}{5}mr^2$ $I_{zz'} = \frac{2}{5}mr^2$ $\text{Volume} = \frac{2}{3}\pi r^3$

Cone		$I_{xx} = \frac{3}{80} m(4r^2 + h^2)$ $I_{yy} = \frac{3}{10} mr^2$ $I_{zz} = \frac{3}{80} m(4r^2 + h^2)$ $I_{xx'} = \frac{1}{20} m(3r^2 + 2h^2)$ $I_{zz'} = \frac{1}{20} m(3r^2 + 2h^2)$ $Volume = \frac{1}{3} \pi r^2 h$
Hallow Shells		
Cylindrical Shell		$I_{xx} = \frac{1}{6} m(3r^2 + h^2)$ $I_{yy} = mr^2$ $I_{zz} = \frac{1}{6} m(3r^2 + h^2)$ <p>* Thickness <math>\ll 1</math></p>
Spherical Shell		$I_{xx} = \frac{2}{3} mr^2$ $I_{yy} = \frac{2}{3} mr^2$ $I_{zz} = \frac{2}{3} mr^2$ <p>* Thickness <math>\ll 1</math></p>

<b>Hemispherical Shell</b>		$I_{xx} = \frac{5}{12} m r^2$ $I_{yy} = \frac{2}{3} m r^2$ $I_{zz} = \frac{5}{12} m r^2$ $I_{xx'} = \frac{2}{3} m r^2$ $I_{zz'} = \frac{2}{3} m r^2$ <p>* Thickness <math>\ll 1</math></p>
Images source: Jacob Moore et al. <a href="http://mechanicsmap.psu.edu/websites/centroidtables/centroids3D/centroids3D.html">http://mechanicsmap.psu.edu/websites/centroidtables/centroids3D/centroids3D.html</a>		

Notice how different objects with the same mass and radius rotate at different rates. This simulation shows a cylinder (blue), ring (green), solid sphere (yellow-brown), and spherical shell (red). Which one has the least inertia? Why?

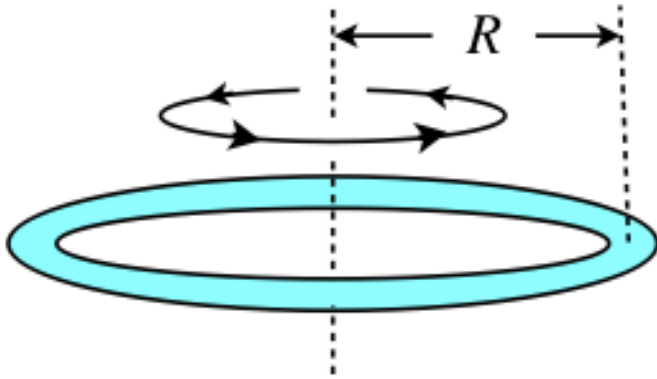


Source: Lucas Vieira.

[https://en.wikipedia.org/wiki/Moment\\_of\\_inertia#/media/File:Rolling\\_Racers\\_-\\_Moment\\_of\\_inertia.gif](https://en.wikipedia.org/wiki/Moment_of_inertia#/media/File:Rolling_Racers_-_Moment_of_inertia.gif)

## 7.4.3 Radius of Gyration

A concept called the *radius of gyration* ( $k$ ) converts a shape into a thin ring. This is used for particularly complex shapes. If a homework problem says ‘the radius of gyration  $k = 15\text{ cm}$ ’, that means if the shape were a thin ring, it would have a radius of  $15\text{ cm}$ . You calculate the mass moment of inertia using the ring equation:  $I = mk^2$



Source: <https://phys.libretexts.org/@go/page/18431>

$$I = mk^2$$

For example, if the mass of an object is  $m=10\text{ kg}$ , the radius of gyration is  $5\text{ m}$ , then the inertia is:

$$I = mk^2 = 10\text{ kg} * 5\text{ m} * 5\text{ m} = 250\text{ kgm}^2.$$

To find the radius of gyration:

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{250\text{ kgm}^2}{10\text{ kg}}} = 5\text{ m}$$

### Key Takeaways

*Basically:* Mass moment of inertia is an object's resistance to rotation and is impacted by mass and distance from the axis of rotation.

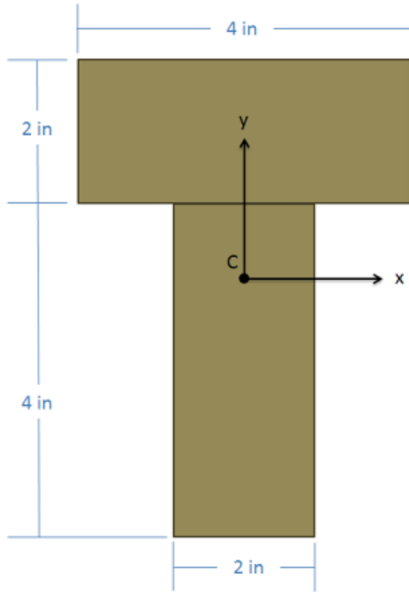
*Application:* The speed that something rotates, such as a satellite spinning in space, is impacted by its inertia. A bigger inertia has a smaller angular acceleration. A smaller inertia allows for a larger angular acceleration.

*Looking Ahead:* This will be used throughout dynamics. The next section looks at calculating inertia of composite objects or from a different axis.

## 7.5 Inertia Intro: Parallel Axis Theorem

There are two great uses for the parallel axis theorem:

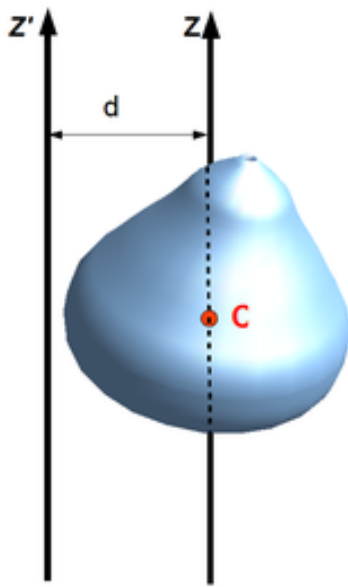
1. Finding the inertia of a complex object with multiple parts.



Source: Jacob Moore et al.  
[mechanicsmap.psu.edu/websites/A2\\_moment\\_integrals/parallel\\_axis\\_theorem/parallelaxistheorem.html](http://mechanicsmap.psu.edu/websites/A2_moment_integrals/parallel_axis_theorem/parallelaxistheorem.html)

2. Rotating an object about an axis other than through the center of mass ( $y'$ )





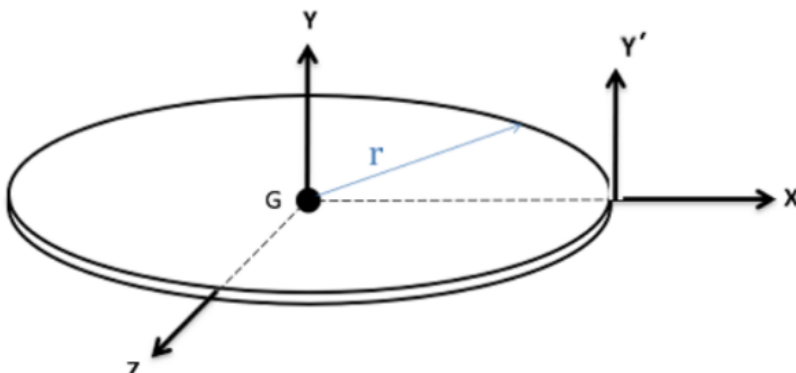
Source (image): By Jack Ver,  
<https://commons.wikimedia.org/w/index.php?curid=6613952>

To begin with, the parallel axis theorem is equal to the inertia about the center of mass ( $I_{cm}$ ) plus the distance between the axes of rotation squared times the mass.

$$I = I_{cm} + md^2$$

### Example 1:

For a disk, the distance between axes  $y$  and  $y'$  is  $d$  and the  $I_{cm} = \frac{1}{2}mr^2$ .



$$I_{yy} = \frac{1}{2}mr^2$$

$$I_{yy'} = \frac{3}{2}mr^2$$

Adapted from: Adapted from source:  
[mechanicsmap.psu.edu/websites/A2\\_moment\\_integrals/parallel\\_axis\\_theorem/parallelaxistheorem.html](http://mechanicsmap.psu.edu/websites/A2_moment_integrals/parallel_axis_theorem/parallelaxistheorem.html)

$$I = I_{cm} + md^2 = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2$$

You will need the table of common geometric shapes in the previous section to find the  $I_{cm}$  for each object.

- $I$  is the moment of inertia of an object with respect to an axis from which the center of mass of the object is a distance  $d$ .
- $I_{cm}$  is the moment of inertia of the object with respect to an axis that is parallel to the first axis and passes through the center of mass.
- $m$  is the mass of the object.
- $d$  is the distance between the two axes.

The parallel axis theorem relates the moment of inertia  $I_{cm}$  of an object, with respect to an axis through the center of mass of the object, to the moment of inertia  $I$  of the same object, with respect to an axis that is parallel to the axis through the center of mass and is at a distance  $d$  from the axis through the center of mass.

A conceptual statement made by the parallel axis theorem is one that you probably could have arrived at by means of common sense, namely that the moment of inertia of an object with respect to an axis through the center of mass is smaller than the moment of inertia about any axis parallel to that one. As you know, the closer the mass is “packed” to the axis of rotation, the smaller the moment of inertia; and, for a given object, per definition of the center of mass, the mass is packed most closely to the axis of rotation when the axis of rotation passes through the center of mass. The PAT is visually shown below, as  $z$  represents the axis on which the object's COM rotates about,  $z'$  is the axis it is now going to rotate about, and  $d$  is the distance between these two axes’.

Source: Calculus-Based Physics 1, Jeffery W. Schnick.

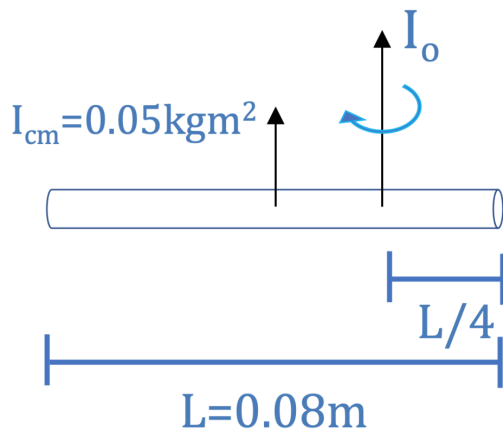
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=ce74a181-ccde-491c-848d-05489ed182e7>

### Steps for finding the MMOI of an object

1. Determine the shape of the object (or shapes, if it a composite object).
2. Determine which axis the object is rotating about.
3. Find the center of mass for each individual shape.
4. Find the  $I_{cm}$  (inertia about its center of mass) for each shape.
5. Determine the distance from the CM of the shapes to the axis of rotation.
6. Use the Parallel Axis Theorem to find the inertia for each shape.
7. Add up all these individual inertias to find  $I_T$ .

### Example 2:

Find the moment of inertia of a uniform rod with  $I_{CM}=0.05\text{kgm}^2$ ,  $L=0.08\text{m}$ , and  $\text{mass}=0.250\text{kg}$ , with respect to an axis that is perpendicular to the rod and passes through at  $1/4$  of the length of the rod.



We know the distance ( $d$ ) to be  $L/4 = 0.08\text{m} / 4 = 0.02\text{m}$  away from the  $z$  axis. Here we present the solution to the problem:

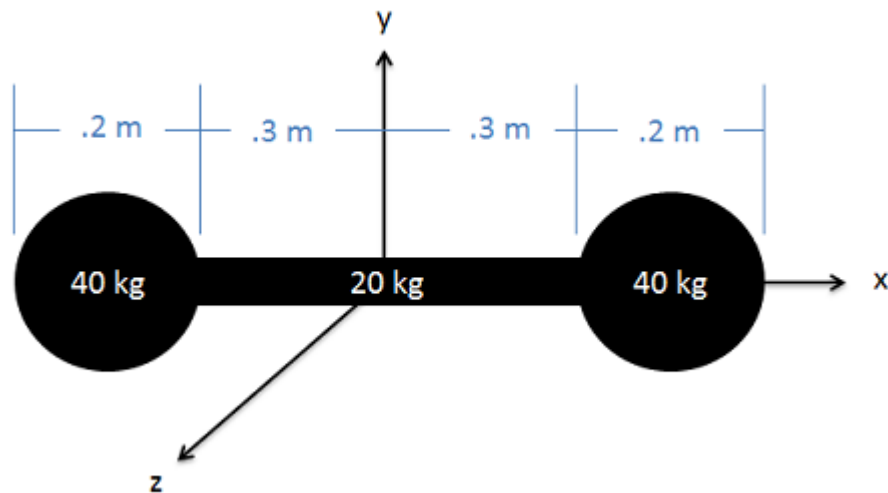
$$I = I_{CM} + md^2$$

$$I = 0.05 \text{ kgm}^2 + (0.250\text{kg})(0.02\text{m})^2$$

$$\underline{I = 0.0501 \text{ kgm}^2}$$

### Example 3

A dumbbell consists of two .2 meter diameter spheres, each with a mass of 40 kg, attached to the ends of a .6 meter long, 20 kg slender rod. Determine the mass moment of inertia of the dumbbell about the  $y$  axis shown in the diagram.



Source:

[mechanicsmap.psu.edu/websites/A2\\_moment\\_integrals/parallel\\_axis\\_theorem/parallel\\_axis\\_theorem.html](http://mechanicsmap.psu.edu/websites/A2_moment_integrals/parallel_axis_theorem/parallel_axis_theorem.html)

Organize the known and unknown data in a table to complete as you go:

	Shape	m	$I_{yyc}$	r	$I_{yyadj}$
x2	1	40 kg	.16 kgm <sup>2</sup>	.4 m	6.56 kgm <sup>2</sup>
	2	20 kg	.6 kgm <sup>2</sup>	0	.6 kgm <sup>2</sup>
	Total	100 kg			13.72 kgm <sup>2</sup>

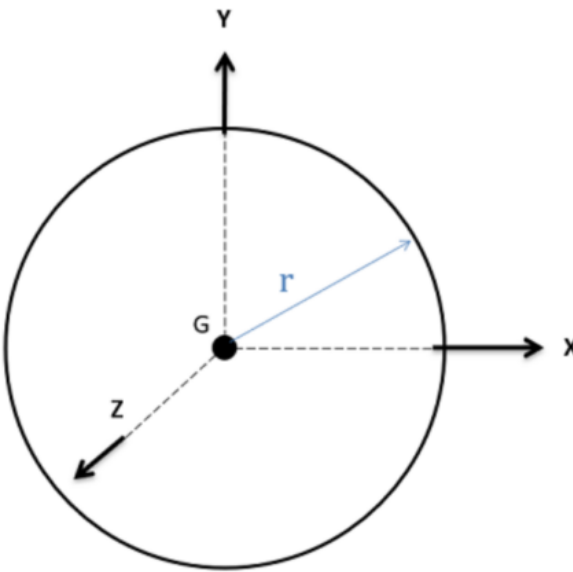
Find the center of mass:

- For each sphere, the  $r_{cm}$  is  $0.3\text{ m} + 1/2 \text{ radius} = 0.3\text{ m} + 1/2 (0.2\text{ m}) = 0.4\text{ m}$
- For the bar, the  $r_{cm}$  is at 0.

Find the inertia about the center of mass for each shape separately. Use the rectangle equation:

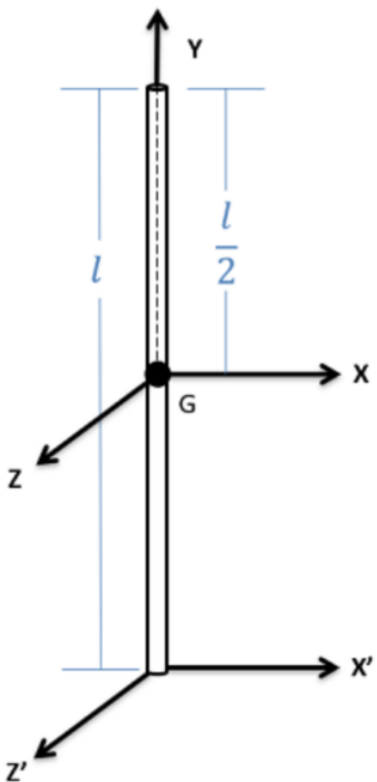
For the sphere:

$$I_{cm-sph} = \frac{2}{5}mr^2 = \frac{2}{5}*(40\text{ kg})*(0.1\text{ m})^2 \quad \quad = 0.16\text{ kgm}^2$$

Sphere		$I_{xx} = \frac{2}{5}mr^2$ $I_{yy} = \frac{2}{5}mr^2$ $I_{zz} = \frac{2}{5}mr^2$ $Volume = \frac{4}{3}\pi r^3$
--------	--	---

For the rod,

$$I_{cm-rod} = \frac{1}{12}ml^2 = \frac{1}{12}*(20\text{kg})*(0.6\text{m})^2 \quad \quad \quad = 0.6 \text{ kgm}^2$$

Slender Rod		$I_{xx} = \frac{1}{12}ml^2$ $I_{yy} = 0$ $I_{zz} = \frac{1}{12}ml^2$ $I_{xx'} = \frac{1}{3}ml^2$ $I_{zz'} = \frac{1}{3}ml^2$ <p>* radius &lt;&lt; length</p>
-------------	---	--

Next the parallel axis theorem is needed to change the axis of rotation from the cm of the sphere to the system cm. The distance between axes of rotation is  $0.3\text{m} + 1/2 \text{ radius} = 0.3\text{m} + 1/2 (0.2\text{m}) = 0.4\text{m}$

$$I_{\text{o-sph}} = I_{\text{cm-sph}} + md^2 = 0.16 \text{ kgm}^2 + (40\text{kg})(0.4\text{m})^2 \quad \quad = 6.56\text{kgm}^2$$

Finally, add up the parts: the 2 spheres and the cm of the rod:

$$I_{\text{total}} = 2 * I_{\text{o-sph}} + I_{\text{cm-rod}} \quad \quad = 2*(6.56 \text{ kgm}^2) + (0.6 \text{ kgm}^2) \quad \quad \underline{I_{\text{total}} = 13.72 \text{ kgm}^2}$$

Source:

[mechanicsmap.psu.edu/websites/A2\\_moment\\_integrals/parallel\\_axis\\_theorem/parallelaxistheorem.html](http://mechanicsmap.psu.edu/websites/A2_moment_integrals/parallel_axis_theorem/parallelaxistheorem.html)

#### Key Takeaways

*Basically:* The parallel axis theorem helps you to find the inertia about a different axis of rotation than the cg, and it lets you combine multiple objects.

*Application:* Find the total inertia of shape with multiple objects.

*Looking Ahead:* This will be used throughout dynamics.

# 7.6 Examples

Here are examples from Chapter 7 to help you understand these concepts better. These were taken from the real world and supplied by FSDE students in Summer 2021. If you'd like to submit your own examples, please send them to the author [eosgood@upei.ca](mailto:eosgood@upei.ca).

## Example 7.6.1: All of Ch 7 - Submitted by William Craine

### 1. Problem

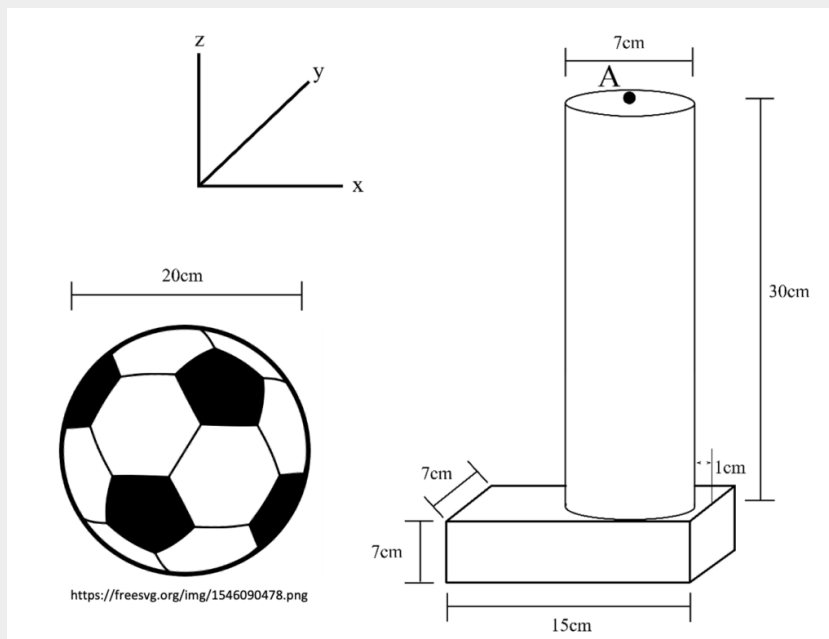
A person is playing soccer. The ball they are using has a diameter of 20 cm, and a mass of 0.45 kg. The person's leg has a mass of 18 kg, and their foot has a mass of 8 kg. Assume that all the shapes are uniform density.

- a) Find the cm for the ball.
- b) Calculate the mass moment of inertia (MMOI) for the ball.
- c) Find the cm for the person's leg and foot.
- d) Find the MMOI for the person's leg and foot on the y-axis about A.



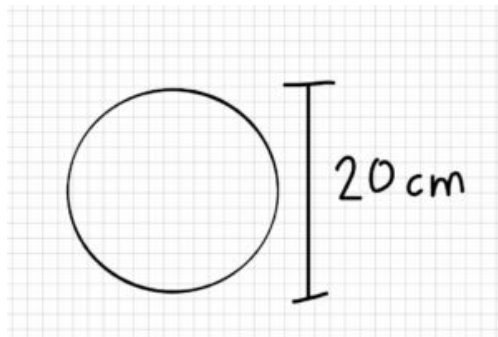
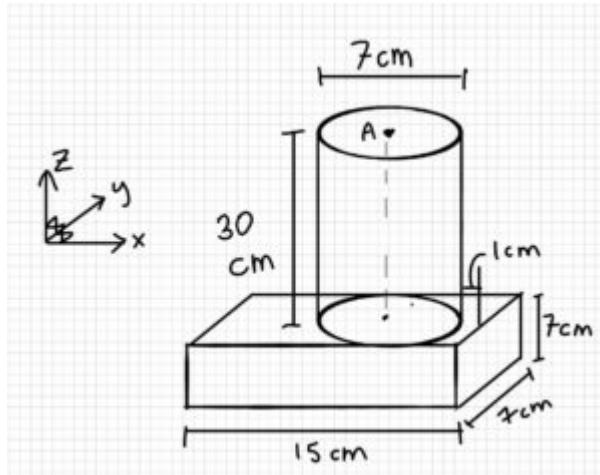
Source:

[https://encrypted-tbn0.gstatic.com/images?q=tbn:ANd9GcTM4e4xHaRSXBdQMGugm1gISi2Qgn7rQx\\_K3w&usqp=CAU](https://encrypted-tbn0.gstatic.com/images?q=tbn:ANd9GcTM4e4xHaRSXBdQMGugm1gISi2Qgn7rQx_K3w&usqp=CAU)



## 2. Draw





### 3. Knowns and Unknowns

Known:

- $m_b = 0.45 \text{ kg}$
- $d_b = 20 \text{ cm}$
- $m_L = 18 \text{ kg}$
- $m_f = 8 \text{ kg}$

Unknowns:  $x_b$ ,  $I_b$ ,  $x_p$ ,  $I_p$

### 4. Approach

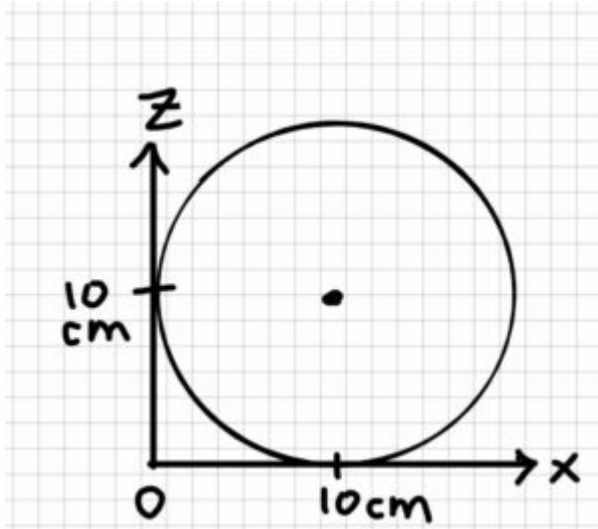
Find the cm for both objects using arbitrary coordinates since no origin is given.

Use the sphere MMOI formula for the ball.

Calculate the individual MMOIs for the leg and foot, then use parallel axis theorem to get each shape's MMOI about the system cm, add them, and then use parallel axis theorem to get MMOI about A.

### 5. Analysis

Part a (find the center of mass of the ball):



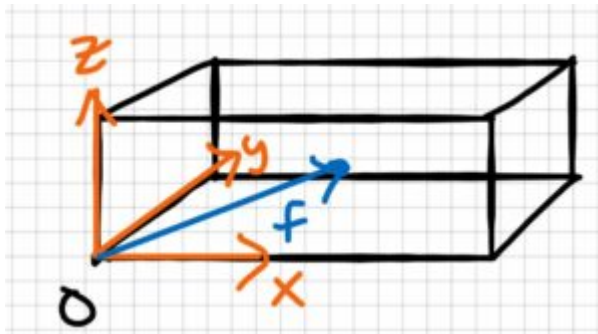
$$\underline{x}_b = (10\hat{i} + 10\hat{j} + 10\hat{k}) \text{ cm}$$

Part b (find the MMOI of the ball about its center of mass):

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{3} m r^2 \quad \quad = \frac{2}{3} (0.45 \text{ kg}) (0.1 \text{ m})^2 \quad I_b = 0.003 \text{ kg m}^2$$

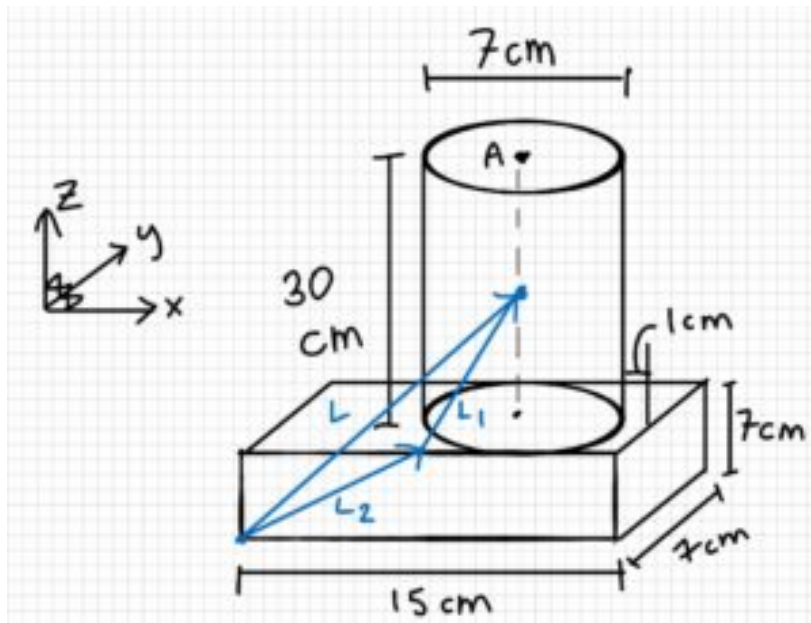
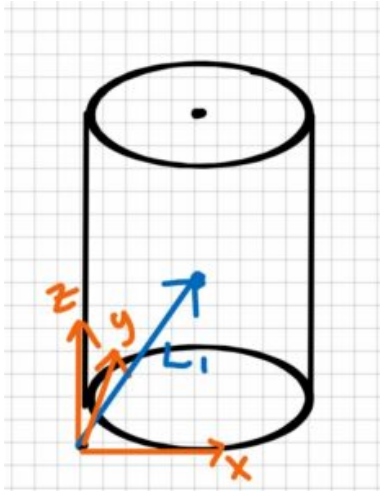
Part c (find the center of mass for the system of the person's leg):

Step 1: find the center of mass of the foot (f)



$$\underline{f} = \frac{15 \text{ cm}}{2} \hat{i} + \frac{7 \text{ cm}}{2} \hat{j} + \frac{7 \text{ cm}}{2} \hat{k} \\ \underline{f} = (7.5\hat{i} + 3.5\hat{j} + 3.5\hat{k}) \text{ cm}$$

Step 2: find the center of mass of the leg (L)



$$\underline{L_1} = (3.5\hat{i} + 3.5\hat{j} + 15\hat{k})\text{cm} \quad \underline{L_2} = (15\text{cm} - 7\text{cm} - 1\text{cm})\hat{i} + (0\text{cm})\hat{j} + (7\text{cm})\hat{k} \quad \underline{L} = (7\hat{i} + 0\hat{j} + 7\hat{k})\text{cm}$$

$$\underline{L} = \underline{L_1} + \underline{L_2} = (3.5\text{cm} + 7\text{cm})\hat{i} + (3.5\text{cm} + 0\text{cm})\hat{j} + (15\text{cm} + 7\text{cm})\hat{k} = (10.5\hat{i} + 3.5\hat{j} + 22\hat{k})\text{cm}$$

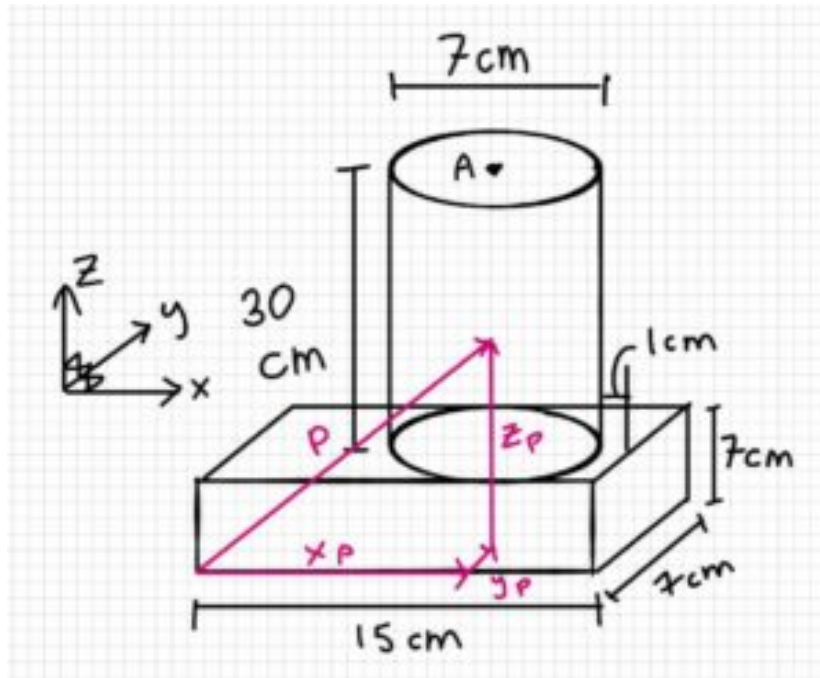
Step 3: find the center of mass of the system (P)

$$x_p = \frac{\sum m_i x_i}{\sum m_i} \quad X_p = \frac{m_f \cdot x_f + m_L \cdot x_L}{m_f + m_L} \quad x_p = \frac{8\text{kg} \cdot 7.5\text{cm} + 18\text{kg} \cdot 10.5\text{cm}}{8\text{kg} + 18\text{kg}} \quad x_p = 9.58\text{cm}$$

$$y_p = \frac{\sum m_i y_i}{\sum m_i} \quad y_p = \frac{8\text{kg} \cdot 3.5\text{cm} + 18\text{kg} \cdot 3.5\text{cm}}{8\text{kg} + 18\text{kg}} \quad y_p = 3.5\text{cm}$$

$$z_p = \frac{\sum m_i z_i}{\sum m_i} \quad z_p = \frac{8\text{kg} \cdot 3.5\text{cm} + 18\text{kg} \cdot 22\text{cm}}{8\text{kg} + 18\text{kg}} \quad z_p = 16.3\text{cm}$$

$$\underline{\underline{P}} = (9.58\hat{i} + 3.5\hat{j} + 16.3\hat{k})\text{cm}$$



Part d (find the inertia of the person's leg about point A):

Step 1: Find the MMOI of the foot about  $f$  ( $I_{ff}$ )

$$x_f = 0.15\text{m} \quad z_f = 0.07\text{m} \quad m_f = 8\text{kg}$$

$$I_{ff} = \frac{1}{12} m \cdot d^2$$

$$(x^2 + z^2) \quad I_{ff} = \frac{1}{12} (8\text{kg})(0.15\text{m}^2 + 0.07\text{m}^2) \quad I_{ff} = 0.0182\text{kg} \cdot \text{m}^2$$

Step 2: Find the MMOI of the foot about  $P$  ( $I_{pf}$ )

$$d^2 = (0.0208\text{m})^2 + (0.128\text{m})^2$$

$$I_{pf} = I_{ff} + m_f(d^2) \quad I_{pf} = 0.0182\text{kgm}^2 + 8\text{kg}[(0.0208\text{m})^2 + (0.128\text{m})^2] \quad I_{pf} = 0.1527\text{kgm}^2$$

Step 3: Find the MMOI of the leg about  $L$  ( $I_{LL}$ )

$$r = 0.035\text{m} \quad h = 0.3\text{m} \quad m_L = 18\text{kg} \quad I_{LL} = \frac{1}{12} m \cdot d^2 \quad I_{LL} = \frac{1}{12} m \cdot (3r^2 + h^2)$$

$$18\text{kg}(3(0.035\text{m})^2 + (0.3\text{m})^2) \quad I_{LL} = 0.1405\text{kgm}^2$$

Step 4: Find the MMOI of the leg about  $P$  ( $I_{PL}$ )

$$\underline{r}_{LP} = [(9.58 - 10.5)\hat{i} + (3.5 - 3.5)\hat{j} + (16.3 - 22)\hat{k}]\text{cm} \quad \underline{r}_{LP} = (-0.92\hat{i} - 5.7\hat{k})\text{cm} \quad I_{PL} = I_{LL} + m(d^2) \quad I_{PL} = 0.1405\text{kgm}^2 + 18\text{kg}[(0.0092\text{m})^2 + (0.057\text{m})^2] \quad I_{PL} = 0.2005\text{kgm}^2$$

Step 5: Find the MMOI of the entire system about P ( $I_P$ )

$$I_P = I_{PL} + I_{pf} \quad I_P = 0.1527 \text{ kgm}^2 + 0.2005 \text{ kgm}^2 = 0.3532 \text{ kgm}^2$$

Step 6: Find the MMOI of the entire system about A ( $I_A$ )

$$A = (10.5 \hat{i} + 3.5 \hat{j} + 37 \hat{k}) \text{ cm} \quad P = (9.58 \hat{i} + 3.5 \hat{j} + 16.3 \hat{k}) \text{ cm} \quad r_{AP} = P - A \\ r_{AP} = [(9.58 - 10.5) \hat{i} + (3.5 - 3.5) \hat{j} + (16.3 - 37) \hat{k}] \text{ cm} \\ r_{AP} = (-0.92 \hat{i} - 20.7 \hat{k}) \text{ cm}$$

$$I_A = I_P + m(d^2) \quad I_A = 0.3532 \text{ kgm}^2 + (8 \text{ kg} + 18 \text{ kg})[(0.0092 \text{ m})^2 + (0.207 \text{ m})^2] \\ I_A = 1.47 \text{ kgm}^2$$

## 6. Review

It makes sense that the numbers are small, since before the final step, the mass was small, or the distance to the new axis was small.

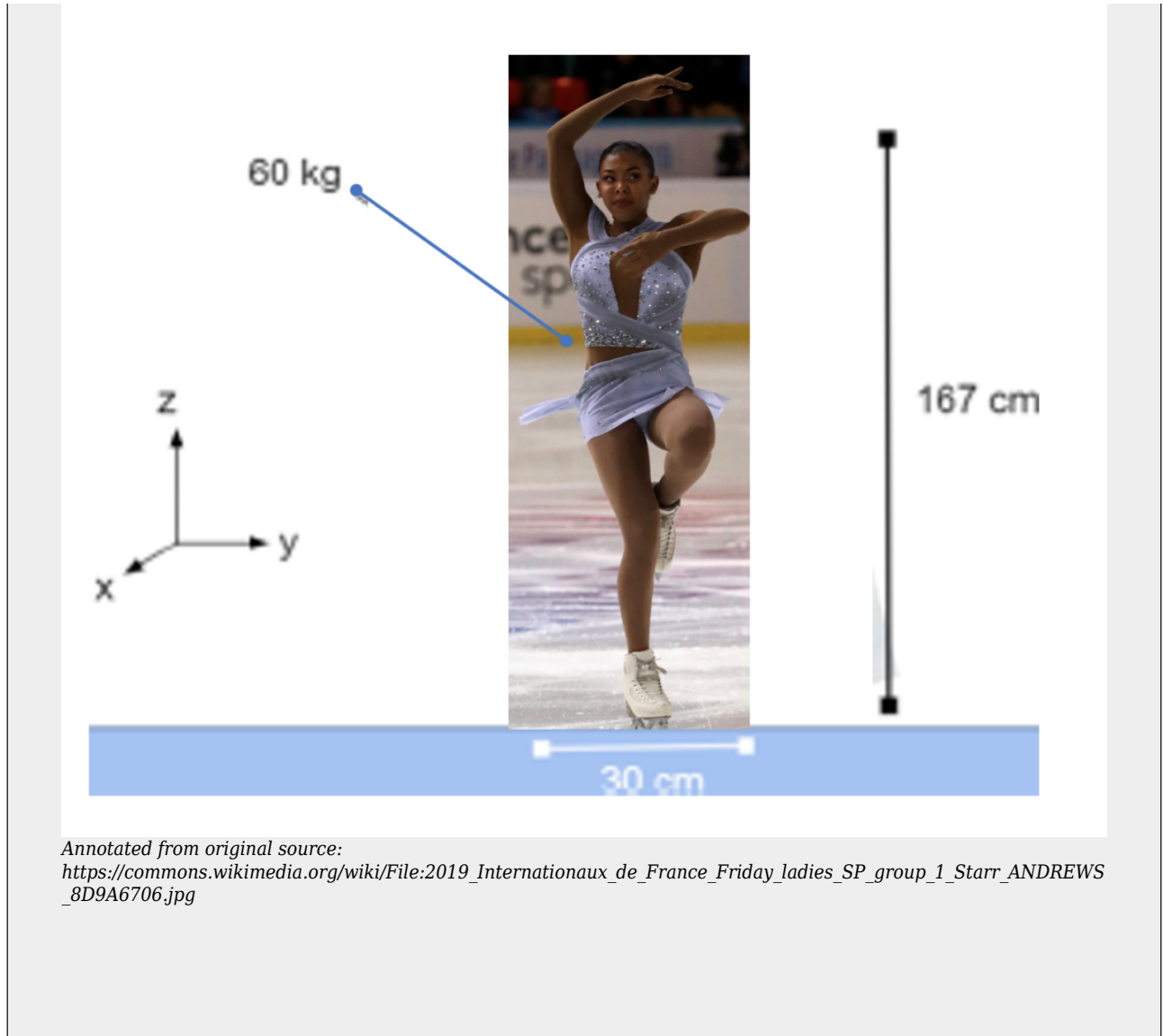
# Example 7.6.2 Inertia - Submitted by Luke McCarvill

## 1. Problem

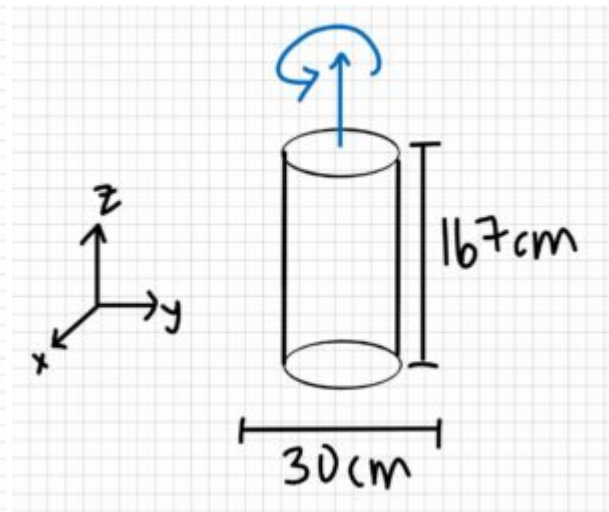
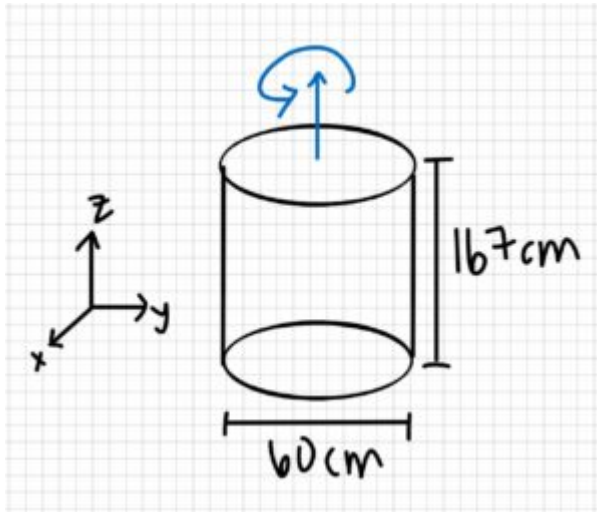
A figure skater with a mass of 60 kg is about to perform a spin about her long axis (z). She is 167 cm tall, and her body can be approximated as a circular cylinder of 30 cm diameter while her limbs are at her side, and a circular cylinder of 60 cm diameter while her arms (and one leg) are outstretched.

a) What should she do in order to generate the highest angular acceleration, assuming she can generate a net torque of 200 Nm? Does lowering her height increase or decrease her angular acceleration?

b) How fast will she be spinning after 0.5 seconds of her maximum vs minimum accelerations, assuming she starts from zero ( $\omega_0 = 0$  rad/sec)?



## 2. Draw



### 3. Knowns and Unknowns

Knowns:

- $m = 60 \text{ kg}$
- $\Sigma M = 200 \text{ Nm}$
- $h = 167 \text{ cm}$
- $d_1 = 30 \text{ cm}$
- $d_2 = 60 \text{ cm}$
- $t = 0.5 \text{ sec}$
- $I_{zz} = \frac{1}{2} mr^2$
- $\Sigma M = I\alpha$
- $\omega_0 = 0 \text{ rad/sec}$

Unknowns:

- $\alpha_1 = ?$
- $\alpha_2 = ?$
- $I_{zz1} = ?$
- $I_{zz2} = ?$
- $\omega_1 = ?$
- $\omega_2 = ?$

### 4. Approach

I'll be using MMOI for circular cylinders, as well as the sum of moments/torque equaling MMOI times angular acceleration, as well as acceleration equaling change in velocity over time.

### 5. Analysis

Part a:

*Step 1: find the inertia when the arms are hugged to the body*

$$I_{zz1} = 0.5 (60 \text{ kg}) (0.15 \text{ m})^2$$

$$I_{zz1} = 0.675 \text{ kg m}^2$$

Step 2: find the inertia when the arms are spread out

$$I_{zz2} = 0.5(60\text{kg})(0.3\text{m})^2$$

$$I_{zz2} = 2.7 \text{ kg m}^2$$

Step 3: find the angular acceleration for both cases

$$\Sigma M = I\alpha \text{ therefore } \Sigma M/I = \alpha$$

$$\alpha_1 = \Sigma M / I_{zz1}$$

$$\alpha_1 = 200 \text{ Nm} / 0.675 \text{ kg m}^2$$

$$\alpha_1 \approx 296.296 \text{ rad/sec}^2$$

$$\alpha_2 = \Sigma M / I_{zz2}$$

$$\alpha_2 = 200\text{Nm}/2.7 \text{ kg m}^2$$

$$\alpha_2 \approx 74.074 \text{ rad/sec}^2$$

The acceleration when the skater had her arms close to her body was about  $296 \text{ rad/sec}^2$ , while that when she had her arms spread out was about  $74 \text{ rad/sec}^2$ . Therefore, **having her limbs closer to her body will give her a much higher angular acceleration!**

As seen in the equation  $I_{zz} = \frac{1}{2} m r^2$ , **her height is arbitrary, thus lowering her height would not change the inertia nor will it change her angular acceleration.**

Part b:

$$\alpha = \Delta\theta/t \text{ therefore } \alpha \cdot t = \Delta\theta$$

$$\Delta\theta_1 = 296.296 \text{ s}^{-2}(0.5\text{s})$$

$$\Delta\theta_1 = 148.148 \text{ rad/sec}$$

$$\Delta\theta_2 = 74.074 \text{ s}^{-2}(0.5\text{s})$$

$$\Delta\theta_2 = 37.037 \text{ rad/sec}$$



Given that these are in radians per second, let's convert this to rotations per second to make it more meaningful. To do so, simply divide by  $2\pi$  since there are  $2\pi$  radians per rotation. Thus, with her arms in, she can achieve about 23.6 rotations per second after 0.5 seconds of acceleration, compared to about 5.9 rotations per second with her limbs out. This is on par with an Olympian according to [this site](#).

## 6. Review

It makes sense that they spin faster when their limbs are hugged to their body; we can try this at home with a swivel chair!

### Engineering Textbooks:

Intro

Engineering:

[https://eng.libretexts.org/Bookshelves/Introduction\\_to\\_Engineering/EGR\\_1010%3A\\_Introduction\\_to\\_Engineering\\_for\\_Engineers\\_and\\_Scientists/14%3A\\_Fundamentals\\_of\\_Engineering/14.11%3A\\_Mechanics/14.11.01%3A\\_Statics](https://eng.libretexts.org/Bookshelves/Introduction_to_Engineering/EGR_1010%3A_Introduction_to_Engineering_for_Engineers_and_Scientists/14%3A_Fundamentals_of_Engineering/14.11%3A_Mechanics/14.11.01%3A_Statics)

Mechanics Map Digital Textbook: <http://www.oercommons.org/courses/mechanics-map-open-mechanics-textbook/view>. Creative Commons Attribution.

Structural

Analysis:

[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames)

Internal Forces in Beams and Frames, Libretexts.  
[https://eng.libretexts.org/Bookshelves/Civil\\_Engineering/Book%3A\\_Structural\\_Analysis\\_\(Udoeyo\)/01%3A\\_Chapters/1.04%3A\\_Internal\\_Forces\\_in\\_Beams\\_and\\_Frames](https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Structural_Analysis_(Udoeyo)/01%3A_Chapters/1.04%3A_Internal_Forces_in_Beams_and_Frames) Creative Commons.

### Physics Textbooks:

University Physics Volume 1: <https://courses.lumenlearning.com/suny-osuniversityphysics/>

Introductory Physics : Building Models to Describe Our World (pdf download):  
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=4c3c2c75-0029-4c9e-967f-41f178bebbbb>

Information at the foundation of modern science and technology from the Physical Measurement Laboratory of NIST: <https://www.physics.nist.gov/cuu/Units/index.html>

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### Math Textbook:

Key Concepts of Intermediate Level Math (pdf download):  
<https://openlibrary.ecampusontario.ca/catalogue/item/?id=d8bdc88b-5439-4652-b4bb-2948f0d5c625>

Calculus Volume 1: <https://openstax.org/books/calculus-volume-1/pages/1-3-trigonometric-functions>

Calculus Based Physics, Jeffrey W. Schnick,

<https://openlibrary.ecampusontario.ca/catalogue/item/?id=ce74a181-ccde-491c-848d-05489ed182e7>

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